Stochastic Growth with Short-run Prediction of Shocks

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Abstract

We study a one sector stochastic growth model with independent and identically distributed shocks where agents acquire information that enables them to accurately predict next period's productivity shock (but not shocks in later periods). Optimal policy depends on the forthcoming shock. We derive conditions under which a more productive realization of the forthcoming shock increases or decreases current investment; relative risk aversion and the elasticity of marginal product play important roles in these conditions. A better shock always increases next period's optimal output if it increases *both* marginal and total product. We derive explicit solutions to the optimal policy function for three well known families of production and utility functions. Volatility of output, sensitivity of output to shocks and expected total invstment may be higher or lower than in the standard stochastic growth model where no new information is acquired over time. Under restrictions similar to that used in the standard model, optimal outputs converge in distribution to a unique invariant distribution whose support is bounded away from zero; the limiting distribution may differ from that obtained in the standard model.

Keywords: Stochastic growth; information; prediction; productivity shocks.

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1 Introduction

Growth and capital accumulation in an economy are affected by fluctuations that are exogenous to economic decision makers. Models of stochastic growth attempt to take into account these exogenous fluctuations, at least partially, in the form of aggregate technology shocks that affect the return from investment. In these models, economic agents make their current consumption and investment decisions on the basis of a commonly known probability distribution of future technology shocks. While the sources of these exogenous shocks are often left unspecified, they are not necessarily external to society. To a significant extent, they emanate from institutional, political or natural environments in which the economy operates. Examples include fluctuations in agricultural productivity caused by meteorological variations, changes in legal and regulatory systems that govern the conduct of business (for instance, the diversion of entrepreneurial talent to rent seeking activities) and alter the cost of non-market inputs[1]. Modern society is characterized by increasing flow of information about these institutional, political or natural factors that, in turn, enable economic agents to make much better predictions about how these exogenous factors are likely to behave in the short run, though they may continue to face high degree of uncertainty about shocks in the distant future. It is important to understand how the economic incentives for capital accumulation respond to better information about, and better prediction of, aggregate "shocks" in the near future, and in particular, how this affects macroeconomic aggregates in the process of economic growth. This paper addresses these issues in a stochastic growth framework.

We consider a variation of the well known model of one sector stochastic optimal growth (Brock and Mirman, 1972) that can also be interpreted as a model of decentralized "equilibrium growth" under technological uncertainty in a competitive representative agent economy. As in much of the stochastic growth literature, we assume that the production shocks are independent and identically distributed over time. In the received version of this model, consumption and investment decisions are made in each period *prior* to the realization of the production shock next period; the latter affects the output resulting from current investment. In our variation of this model, we assume that in each period, before making consumption and investment decisions, agents acquire new information that enables them to predict the realization of the shock *next period*. In order to bring out the effect of short run predictability in a stark fashion, we assume that the prediction is accurate, i.e., agents foresee correctly the *exact* realization of next period's shock. However, the information available does not affect the agents' beliefs about the probability distribution of shocks in later time periods. Thus, in our model, though there is no uncertainty about next period's return on investment, agents remain uncertain about the production technology and therefore, the value of accumulation in later periods.

Though the information structure in our model is a relatively minor modification of that in the standard optimal stochastic growth model, it creates major qualitative differences to the nature of optimal policy. In particular, optimal consumption and investment decisions are now sensitive to the predicted realization of the forthcoming shock.

The main contributions of our paper are as follows. We characterize the sensitivity and qualitative dependence of optimal decisions on the prediction of next period's shock. In particular, when the total and marginal productivity are ordered by the realized shock, we

[1] See, Hansen and Prescott (1993).

examine the effect of "better" shocks on optimal investment and next period's output. While next period's output always increases with a better shock, investment may increase or decrease with a better shock depending on the curvature of the optimal *value* function. For the case of multiplicative shocks, we outline conditions on the utility and production functions that ensures that investment increases or decreases with a better shock. Our results indicate that investment increases (and consumption declines) with a better shock if the degree of relative risk aversion and the elasticities of total and marginal product are above a critical level.

We derive explicit solutions to the optimal policy function for three well-known cases and compare the outcomes of our model to the standard stochastic growth model. These explicit solutions are not only used to illustrate certain qualitative properties, but are also likely to be very useful in future macroeconomic applications.

Though availability of information about next period's shock makes optimal investment and consumption sensitive to the shock, it allows agents to absorb some of the variation in shocks by adjusting their current consumption and investment. As a result, the transmission of volatility of the shocks to next period's output may be higher or lower compared to the standard stochastic growth model. We show this in some specific examples where, depending on parametric conditions, information about the forthcoming shock may magnify or dampen output volatility. Also, depending on parameters, the expected total investment may be higher or lower than in the standard model. Our analysis indicates that information about forthcoming shocks increases the role of the utility function in determining the qualitative nature of economic outcomes as well as comparative dynamics (relative to the standard stochastic growth framework).

Finally, we show that despite the dependence of optimal actions on the forthcoming shock, under very similar restrictions as in imposed in the standard stochastic growth model, the stochastic process of optimal outputs converge in distribution to a unique invariant distribution whose support is bounded away from zero. This unique stochastic steady state itself may, however, differ from that obtained in the standard model; even though the difference in information structure of the two models pertains only to the short run i.e., whether or not one can predict the immediately forthcoming shock, differences in the economic processes generated may persist in the long run.

Our paper is related to several strands of the existing literature. First, there is a large literature on models of real business cycles where cyclical fluctuations are related to imperfect forecasting of future productivity shocks by agents that observe signals that are correlated with future shocks. While the idea goes back to Pigou (1927), much of the literature is fairly recent where the focus is on explaining specific features of observed cycles including booms and recessions, persistence of macroeconomic aggregates and co-movement in output, investment and consumption.[2] The basic model used in this literature is the neoclassical stochastic growth model and as in our paper, agents observe signals, albeit imperfect, of future shocks; in fact, our structure can be seen as one where the signal about next period's shock is fully informative. However, there are significant differences. We do not seek to generate cycles or explain any of the observed empirical regularities in the business cycles literature. Unlike models of business cycles where shocks are serially correlated, we assume that productivity shocks are i.i.d. over time. Our focus is on understanding capital accumulation and much of

[2]See, among others, important contributions by Danthine, Donaldson and Johnsen (1998), Beaudry and Portier (2004, 2007), Schmitt-Grohe and Uribe (2008) and Jaimovich and Rebelo (2009),

our analysis is carried out in a general framework.

The second strand of literature that relates to our paper is the one that analyzes experimentation and learning in a stochastic growth model[3]. In this literature, agents acquire (actively or passively) signals over time that enable them to learn about unknown parameters affecting the production function or the distribution of shocks in a Bayesian fashion. Unlike this literature, in our case there is no imperfect information about the structure of the economy; in particular, the initial condition, the production function and the distribution of the i.i.d. shocks are fully known. The paper does not address the question of structural learning. In our model, agents acquire new information that allows perfect prediction of the realization of next period's shock, but this does not, in any way, alter their posterior distribution of shocks in later periods.

The third strand of literature related to our framework is the one that examines the effects of 'better information' (Blackwell, 1953) on the behavior of economic agents and their aggregate implications for dynamic equilibrium. In an overlapping generation model with investment in human capital, Eckwert and Zilcha (2004) show that better information may either enhance, or reduce, the aggregate stock of human capital along the equilibrium path, depending on the risk aversion parameters. The motivation for our paper is best viewed in terms of this effect of better information that enables agents predict the realization of economy-wide shocks in the short run future.

Finally, within the optimal stochastic growth literature, Donaldson and Mehra (1983) analyze a general one sector model of stochastic growth with correlated shocks. In their framework, past realizations of productivity shocks allow agents to update the posterior distribution of all future shocks. As mentioned earlier, shocks are independent in our framework and previous shocks carry no information about future shocks.

The results in our paper have applications in other areas such as optimal management of biological species (and other renewable resources) under uncertainty about their "natural growth". In particular, it allows us to understand the implication of being able to correctly forecast short run environmental conditions that cause fluctuations in the growth of populations of biological species.

Our paper is organized as follows. Section 2 describes the model and contains some basic results on existence and policy functions. In Section 3, we outline three well-known families of utility and production functions for which we explicitly derive analytical solutions to the optimal investment and consumption policy. In Section 4, we analyze the monotonicity of output, investment and consumption in (the predicted) realizations of forthcoming productivity shock. Section 5 discusses the effect of information about forthcoming productivity shocks and the ability to predict their realizations by comparing the dynamic optimal policy of our model to that in the standard stochastic growth model; in particular, we discuss the effect of information on investment, output, sensitivity of output to the shock and the volatility of output. Section 6 discusses long run convergence properties. Almost all formal proofs are relegated to the Appendix.

[3]See, among others, Freixas (1981), Demers (1991), Mirman, Samuelson and Urbano (1993), and Koulovatianos, Mirman and Santugini (2009). Nyarko and Olson (1996) study a version of the model with imperfect information and learning about the capital stock. See, also Majumdar (1982).

2 Preliminaries.

We consider an infinite horizon one-good representative agent economy. Time is discrete and is indexed by $t = 0, 1, 2, ...$At each date $t \geq 0$, the representative agent observes current output y_t as well as (an accurate prediction of) the realization of ρ_{t+1}, the random production shock that affects the production function at the beginning period $(t+1)$; the shocks are independent over time so that the realization of ρ_{t+1} provides no additional information about technology shocks in periods $\tau > t+1$. After this, the agent chooses the level of current investment x_t, and the current consumption level c_t, such that

$$c_t \geq 0, x_t \geq 0, c_t + x_t \leq y_t$$

This generates y_{t+1}, the output next period through the relation

$$y_{t+1} = f(x_t, \rho_{t+1})$$

where $f(.,.)$ is the "aggregate" production function. The economy begins with a given initial stock of output $y_0 > 0$ and a given (accurate prediction of) the realization of ρ_1. The capital stock depreciates fully every period. Given current output $y \geq 0$, the feasible set for consumption and investment is denoted by $\Gamma(y)$ i.e.,

$$\Gamma(y) = \{(c, x) : c \geq 0, x \geq 0, c + x \leq y\}$$

Note that the prediction of next period's shock does not affect the feasible set of consumption and investment in the current period.

The following assumption is made on the sequence of random shocks:

(A.1) $\{\rho_t\}_{t=1}^{\infty}$ *is an independent and identically distributed random process defined on a probability space* $(\Omega, \mathcal{F}, P)$*, where the marginal distribution function is denoted by* F*. The support of this distribution is a compact set* $A \subset \mathbb{R}$*.*

The production function f is assumed to satisfy the following:

(T.1) *For all* $\rho \in A, f(x, \rho)$ *is concave in* x *on* $\mathbb{R}_+$*.*

(T.2) *For all* $\rho \in A, f(0, \rho) = 0$*.*

(T.3) *For each* $\rho \in A, f(x, \rho)$ *is continuously differentiable in* x *on* $\mathbb{R}_{++}$ *and, further,* $f'(x, \rho) = \frac{\partial f(x,\rho)}{\partial x} > 0$ *on* $\mathbb{R}_{++} \times A$*.*

(T.4) $\inf_{\rho \in A}[\lim_{x \to 0} f'(x, \rho)] > 1$.

Assumptions (**T.1**)-(**T.3**) are standard monotonicity, concavity and smoothness restrictions on production. **(T.4)** ensures that the technology is productive with probability one in a neighborhood of zero. Note that we do not require that the production functions be ordered in the realization of the random shock though we will make that assumption in a later section.

Let $\beta \in (0, 1)$ denote the time discount factor. Given the initial stock $y_0 > 0$, the representative agent's objective is to maximize the discounted sum of expected utility from consumption:

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

where u is the one period utility function from consumption.

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. The utility function $u : \mathbb{R}_+ \to \overline{\mathbb{R}}$ satisfies the following restrictions:

(U.1) *u is strictly increasing, continuous and strictly concave on* $\mathbb{R}_+$ (on $\mathbb{R}_{++}$ if $u(0) = -\infty$); $u(c) \to u(0)$ as $c \to 0$.

(U.2) *u is twice continuously differentiable on* $\mathbb{R}_{++}; u'(c) > 0, u''(c) < 0, \forall c > 0$.

(U.3) $\lim_{c \to 0} u'(c) = +\infty$.

Assumptions **(U.1)** and **(U.2)** are standard. Note that we allow the utility of zero consumption to be $-\infty$. **(U.3)** requires that the utility function satisfy the Uzawa-Inada condition at zero and ensures that optimal consumption and investment lie in the interior of the feasible set.

The *partial history* at date t is given by $h_t = (y_0, \rho_1, x_0, c_0, \ldots, y_{t-1}, \rho_t, x_{t-1}, c_{t-1}, y_t, \rho_{t+1})$. A *policy* π is a sequence $\{\pi_0, \pi_1, \ldots\}$ where π_t is a conditional probability measure such that $\pi_t(\Gamma(y_t)|h_t) = 1$. A policy is *Markovian* if for each t, π_t depends only on (y_t, ρ_{t+1}). A Markovian policy is *stationary* if π_t is independent of t. Associated with a policy π and an initial state (y, ρ) is an expected discounted sum of social welfare:

$$V_\pi(y, \rho) = E \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $\{c_t\}$ is generated by π, f in the obvious manner and the expectation is taken with respect to P.

The *value function* $V(y, \rho)$ is defined on $\mathbb{R}_{++} \times A$ by:

$$V(y, \rho) = sup\{V_\pi(y, \rho) : \pi \textit{ is a policy}\}.$$

Under assumption **(T.4)**, it is easy to check that

$$-\infty < V(y, \rho), \forall y > 0, \rho \in A.$$

We will assume that:

(V.1) $V(y, \rho) < +\infty$, $\forall y > 0, \rho \in A$..

It is easy to check that **(V.1)** is satisfied if the technology exhibits bounded growth i.e., there exists $K > 0$ such that $\frac{f(x,\rho)}{x} < 1$ for all $x > K$ and for all $\rho \in A$. Even if the technology allows for unbounded expansion of consumption, **(V.1)** is satisfied if the utility function is bounded above or, alternatively, the discount factor is small enough (smaller than an asymptotic growth factor).

A policy, π^*, is *optimal* if $V_{\pi^*}(y, \rho) = V(y, \rho)$ for all $y > 0, \rho \in A$. Standard dynamic programming arguments imply that there exists a *unique* optimal policy, that this policy is stationary and that the value function $V(y, \rho)$ satisfies the functional equation:

$$V(y, \rho) = \sup_{x \in \Gamma(y)} [u(y - x) + \beta E_{\rho'}[V(f(x, \rho), \rho')], y > 0, \rho \in A. \tag{1}$$

In the functional equation (1), ρ is next period's shock affecting the output from current investment whose realization is predicted (correctly) prior to deciding on current consumption and investment, while ρ' is the shock that will affect the production function two periods later (and whose realization, though unknown now, will be predicted accurately next period); the expectation on the right hand side of (1) is taken with respect to the random variable ρ'.

It can be shown that for any $\rho \in A$, $V(y, \rho)$ is continuous, strictly increasing and strictly concave in y on $\mathbb{R}_{++}$. Further, the maximization problem on the right hand side of (1) has a unique solution, denoted by $x(y,\rho)$. The stationary policy generated by the function $x(y,\rho)$ is the optimal policy and we refer to $x(y,\rho)$ as the optimal investment function. $c(y,\rho) = y - x(y,\rho)$ is the optimal consumption function. Using small variations of the standard arguments in the literature, **(U.3)** can be used to show that:

Lemma 1 *For all $y > 0, \rho \in A, x(y,\rho) > 0$ and $c(y,\rho) > 0$.*

Further,

Lemma 2 *For all $\rho \in A, x(y,\rho)$ and $c(y,\rho)$ are continuous and strictly increasing in y on $\mathbb{R}_{++}$.*

Using identical arguments to that in Mirman and Zilcha (1975), we have:

Lemma 3 *$V(y,\rho)$ is differentiable in y on $\mathbb{R}_{++}$ and it satisfies:*

$$V'(y,\rho) = u'[c(y,\rho)] \quad \text{for all} \quad y > 0, \tag{2}$$

where $V'(y,\rho)$ denotes the partial derivative of V with respect to its first argument.

Finally, we note that the following version of the stochastic Ramsey-Euler equation holds:

Lemma 4 *For all $y > 0, \rho \in A$*

$$u'(c(y,\rho)) = \beta f'(x(y,\rho),\rho) E_{\rho'}[u'(c(f(x(y,\rho),\rho')))]. \tag{3}$$

Observe that unlike the standard stochastic growth model, for any given $y > 0$, (3) is required to hold for every possible realization ρ of the forthcoming shock. The term $\beta f'(x(y,\rho),\rho)$ on the right hand side of (3) captures the marginal productivity of investment which is deterministic (given ρ), while the term $E_{\rho'}[u'(c(f(x(y,\rho),\rho')))]$ captures the future expected marginal valuation of the additional output created through investment; the marginal valuation is stochastic because it depends on next period's consumption which is influenced by the (yet unknown) random shock ρ' of the period after next.

Assumptions (A.1), (T.1) - (T.4) and (V.1) hold throughout the paper. Lemmas and propositions will specifically mention all additional assumptions.

3 Optimal Policy: Explicit Solutions.

In this section, we outline three well-known families of utility and production functions for which we explicitly derive analytical solutions to the optimal investment and consumption policy functions. We will use these later to illustrate the nature of dependence of macroeconomic aggregates on predicted shocks and to illustrate the effect of information about forthcoming shocks by comparing them to optimal policies derived in the standard stochastic growth model with no prior information about forthcoming shocks.

3.1 CES Utility and Linear Production Function.

In this subsection, we consider an economy where the production function is linear and the utility function exhibits constant elasticity of substitution (or, constant relative risk aversion). In particular:

$$
\begin{aligned}
u(c) &= \frac{c^{1-\sigma}}{1-\sigma}, \qquad \sigma \neq 1, \sigma > 0. &(4)\\
&= \ln c, \quad \sigma = 1. &(5)
\end{aligned}
$$

$$f(x,\rho) = \rho x \tag{6}$$

where

$$\underline{\rho} = \inf A > 1. \tag{7}$$

In the existing stochastic growth literature, the linear production technology was first analyzed by Levhari and Srinivasan (1969), and this particular family of utility and stochastic production functions has been extensively used in the literature on unbounded stochastic growth (see for example, De Hek, 1999).

We also impose the restriction :

$$\beta E(\rho^{1-\sigma}) < 1. \tag{8}$$

which ensures the existence of an optimal policy.

Recall that $c(y,\rho), x(y,\rho)$ denote the optimal consumption and investment functions. Let,

$$y'(y,\rho) = f(x(y,\rho),\rho) = \rho x(y,\rho).$$

From (3):

$$(c(y,\rho))^{-\sigma} = \beta\rho E_{\rho'}[(c(y'(y,\rho),\rho'))^{-\sigma}] \tag{9}$$

We conjecture that optimal policy function is linear in current output:

$$c(y,\rho) = \lambda(\rho)y$$

Then,

$$y'(y,\rho) = \rho(1-\lambda(\rho))y$$

$$c(y'(y,\rho),\rho') = \lambda(\rho')\rho(1-\lambda(\rho))y$$

Thus, (9) can be re-written as:

$$
\begin{aligned}
(\lambda(\rho)y)^{-\sigma} &= \beta\rho E_{\rho'}[(\lambda(\rho')\rho(1-\lambda(\rho))y)^{-\sigma}]\\
&= \beta\rho^{1-\sigma}((1-\lambda(\rho))y)^{-\sigma}E_{\rho'}[(\lambda(\rho'))^{-\sigma}]
\end{aligned}
$$

which can be re-written as:

$$\frac{(\lambda(\rho))^{-\sigma}}{\beta\rho^{1-\sigma}(1-\lambda(\rho))^{-\sigma}} = E_{\rho'}[(\lambda(\rho'))^{-\sigma}]$$

Let

$$\mu = E_{\rho'}[(\lambda(\rho'))^{-\sigma}]$$

Then,

$$\frac{(\lambda(\rho))^{-\sigma}}{\beta\rho^{1-\sigma}(1-\lambda(\rho))^{-\sigma}} = \mu$$

so that :

$$\lambda(\rho) = \frac{1}{1+(\mu\beta)^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}}} \tag{10}$$

and the optimal policy functions are given by:

$$c(y,\rho) = [\frac{1}{1+(\mu\beta)^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}}}]y \tag{11}$$

$$x(y,\rho) = [\frac{(\mu\beta)^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}}}{1+(\mu\beta)^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}}}]y, \tag{12}$$

where the constant μ is implicitly determined by:

$$\mu = E[(\lambda(\rho))^{-\sigma}] = E[(1+(\mu\beta)^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}})^{\sigma}] \tag{13}$$

i.e.,

$$E[(\mu^{-\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}})^{\sigma}] = 1 \tag{14}$$

Note that the left hand side of (14) is strictly decreasing in μ and diverges to $+\infty$ as $\mu \to 0$. Further, using (8), the left hand side of (14) converges to $\beta E(\rho^{1-\sigma}) < 1$ as $\mu \to +\infty$. Thus, there exists unique $\mu > 0$ that solves (14). Further, from (13), we can see that

$$\mu > 1.$$

Suppose $\sigma = 1$. Then, (14) implies $\mu = \frac{1}{1-\beta}$and

$$c(y,\rho) = (1-\beta)y, x(y,\rho) = \beta y$$

which is independent of ρ. In other words, with a linear technology and logarithmic utility, the optimal policy function is independent of the shocks (and in fact, identical to that obtained in the standard model where agents invest and consume without observing the forthcoming shock).

For $\sigma \neq 1$, one cannot solve for the constant μ explicitly. However, one can obtain considerable information from the implicit equation (14) defining μ. Observe that optimal consumption $c(y,\rho)$ is decreasing (increasing) and optimal investment $x(y,\rho)$ is increasing (decreasing) in the productivity shock ρ if the relative risk aversion σ is greater than (less than) one. In other words, whether a (predicted) productivity increasing realization of shock increases or reduces investment depends on preferences and, in particular, relative risk aversion (or, the intertemporal elasticity of substitution). Observe, however,

$$\begin{aligned} y'(y,\rho) &= \frac{\rho(\mu\beta)^{\frac{1}{\sigma}}}{\rho^{1-\frac{1}{\sigma}} + (\mu\beta)^{\frac{1}{\sigma}}}y \\ &= \frac{(\mu\beta)^{\frac{1}{\sigma}}}{\rho^{-\frac{1}{\sigma}} + \rho^{-1}(\mu\beta)^{\frac{1}{\sigma}}}y \end{aligned}$$

which is strictly increasing in ρ for all $\sigma > 0$. In other words, independent of the level of risk aversion, the output next period is always increasing in the productivity shock. We will see later that these are more general properties for technologies ordered by the random shock.

Note that the above policy functions have been derived by using the Ramsey-Euler equation. To show that they are optimal, we need to verify that the transversality condition is also satisfied i.e., $\beta^t EV'(y_t^*, \rho_t) \to 0$ where $\{y_t^*\}$ is the stochastic process of output generated by the optimal policy, given $y_0^* = y_0$ and given ρ_1. This is contained in the appendix.

3.2 Log Utility and Cobb-Douglas production function with exponential shock.

In this subsection, we consider the economy where the utility function is logarithmic:

$$u(c) = \ln c$$

and the production function is Cobb-Douglas exhibiting bounded growth.

$$f(x, \rho) = \ x^{\rho},$$

where

$$0 < \underline{\rho} = \inf A \leq \overline{\rho} = \sup A < 1.$$

Note that the random shock is not multiplicative but rather affects the exponent of the Cobb-Douglas production function. In the existing stochastic growth literature where agents acquire no prior information about future shocks, explicit solution for the optimal policy function in this economy was obtained by Mirman and Zilcha (1975). Note that $f(x, \rho)$ is decreasing in ρ on $[0, 1]$ and increasing in ρ for $\rho \geq 1$. Further, $f(x, \rho) < x$ for all $x > 1$ with probability one, so that given initial conditions, all possible consumption and investment paths are uniformly bounded. Thus, **(V.1)** is satisfied.

To obtain the optimal policy function, we conjecture that the unique optimal consumption function is linear in output and has the form:

$$c(y, \rho) = \lambda(\rho)y.$$

The Ramsey-Euler (3) then implies:

$$\frac{1}{\lambda(\rho)y} = \beta A\rho[(1 - \lambda(\rho))y]^{\rho-1} E_{\rho'}\{\frac{1}{\lambda(\rho')A[(1 - \lambda(\rho))y]^{\rho}}\}$$

which yields:

$$\frac{1}{\lambda(\rho)} = \beta\rho\frac{1}{1 - \lambda(\rho)}E[\frac{1}{\lambda(\rho')}]$$

Let $\widehat{m} = E\{[\lambda(\rho')]^{-1}\}$. Then,

$$\frac{1}{\lambda(\rho)} = 1 + \beta\rho\widehat{m}$$

and taking the expectation on both sides with respect to ρ we have:

$$\widehat{m} = \frac{1}{1 - \beta E[(\rho]} \tag{15}$$

which implies

$$\lambda(\rho) = \frac{1}{1 + \beta\rho[1 - \beta E(\rho)]^{-1}} = \frac{1 - \beta E(\rho)}{1 + \beta[\rho - E(\rho)]} \tag{16}$$

which is a decreasing function of ρ. Observe that $E(\rho) < 1$ and so, $0 < \lambda(\rho) < 1$. The optimal policy functions are given by:

$$c(y, \rho) = [\frac{1 - \beta E(\rho)}{1 + \beta[\rho - E(\rho)]}]y \tag{17}$$

$$x(y, \rho) = [\frac{\beta\rho}{1 + \beta[\rho - E(\rho)]}]y, \tag{18}$$

The transversality condition is easily verified as feasible paths are uniformly bounded. Observe that in this case, optimal consumption is always decreasing in ρ and optimal investment is, therefore, increasing in ρ. However, as we shall see later, optimal output next period is not necessarily monotonic in ρ.

3.3 Log Utility and Cobb-Douglas production function with multiplicative shock.

In this subsection, we consider the economy where the utility function is logarithmic as before:

$$u(c) = \ln c$$

and the production function is Cobb-Douglas with multiplicative shock:

$$f(x, \rho) = \rho\ x^{\theta}, 0 < \theta < 1.$$

We assume that

$$0 < \underline{\rho} = \inf A.$$

For the standard stochastic growth model, explicit solution for the optimal policy function for this case is contained in Mirman and Zilcha (1975). Note that the production function is increasing in the shock. Further, given initial conditions, all possible consumption and investment paths are uniformly bounded. Thus, **(V.1)** is satisfied.

To obtain the optimal policy function for our model, we conjecture that the unique optimal consumption function is linear in output and has the form:

$$c(y, \rho) = \lambda(\rho)y.$$

Then, from (3):

$$\begin{aligned}
\frac{1}{\lambda(\rho)y} &= \beta\rho\theta[(1 - \lambda(\rho))y]^{\theta-1}E_{\rho'}\{\frac{1}{\lambda(\rho')\rho[(1 - \lambda(\rho))y]^{\theta}}\} \\
&= \beta\theta[(1 - \lambda(\rho))y]^{-1}E_{\rho'}\{\frac{1}{\lambda(\rho')}\}
\end{aligned}$$

which yields:

$$\frac{1}{\lambda(\rho)} = \beta\theta \frac{1}{1-\lambda(\rho)} E[\frac{1}{\lambda(\rho')}]$$

Let $\widehat{m} = E\{[\lambda(\rho')]^{-1}\}$. Then,

$$\frac{1}{\lambda(\rho)} = 1 + \beta\theta\widehat{m}$$

and taking the expectation on both sides with respect to ρ we have:

$$\widehat{m} = \frac{1}{1-\beta\theta}$$

which implies

$$\lambda(\rho) = \frac{1}{1+\beta\theta[1-\beta\theta]^{-1}} = \beta\theta.$$

which is independent of ρ. Observe that $\lambda(\rho) \in (0,1)$. The optimal policy functions are given by:

$$\begin{aligned} c(y,\rho) &= \beta\theta y \\ x(y,\rho) &= (1-\beta\theta)y. \end{aligned}$$

The transversality condition is easily verified as feasible paths are uniformly bounded. Observe that in this case, optimal consumption and optimal investment are independent of ρ. The optimal policy functions are in fact identical to that obtained in the conventional model with no information about forthcoming shocks. In other words, information about impending shock has no impact on the economy! As the production function is increasing in ρ, the optimal output next period is always increasing in ρ.

4 Effect of More Productive Realizations of Shocks.

The primary purpose of this paper is to understand the manner in which the ability to predict forthcoming technology shocks (in the short run) affects macroeconomic aggregates and behavior. In this section, we analyze the monotonicity of output, investment and consumption in (the predicted) realizations of forthcoming productivity shock. For this analysis to be meaningful, it makes sense to confine attention to production functions that are ordered by the realizations of the shock. Therefore, we assume that the total and the marginal product resulting from any level of investment are increasing in the shocks. In that case, higher values of the realizations of the productivity shock can be interpreted as "better" or, more productive. For simplicity of analysis we will also assume that the production function is smooth in investment and shocks, and that the support of the distribution of shocks is an interval.

(T.5) The support A of the distribution F of productivity shocks is a interval $[\underline{\rho}, \overline{\rho}] \subset \mathbb{R}$. $f(x,\rho)$ is twice continuously differentiable on $\mathbb{R}_{++} \times A$. Further, for any $x > 0, \rho \in A$, $\frac{\partial f}{\partial \rho} > 0$, $\frac{\partial^2 f}{\partial \rho \partial x} > 0$.

Assumption **(T.5)** is retained throughout this section.

4.1 Effect of Better Shocks on Investment and Consumption.

We begin by analyzing how investment and consumption change when the predicted realization of the forthcoming shock is better. In particular, under assumption **(T.5)** higher values of ρ indicate higher total and marginal return on investment. We examine the conditions under which optimal investment $x(y,\rho)$ is increasing in ρ (i.e., optimal consumption $c(y,\rho)$ is decreasing in ρ).

Economic intuition suggests that there are two effects when an agent foresees a better realization of next period's shock First, there is an increase in the incentive to invest as the return on investment is higher. Second, there is an increase in the incentive to increase current consumption because a lower level of investment is enough to generate the same output next period as would have been optimal if the observed shock was less productive. Which effect dominates should depend on the intertemporal elasticity of substitution that, in this model, is simply the inverse of relative risk aversion.

The clearest illustration of this basic intuition is obtained by looking at the specific economy discussed in Section 3.1 where the production function is linear (given by (6) and (7)), the utility function exhibits constant elasticity of substitution (given by (4) and (5)). Under assumption (8), the explicit form of the optimal policy functions for this economy are given by (11), (12) and (14). In particular, from (11) we have that the optimal investment policy function is given by:

$$x(y,\rho) = [\frac{(\mu\beta)^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}}}{1+(\mu\beta)^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}}}]y \tag{19}$$

where $\mu > 1$ is a constant (defined implicitly) and σ is the (constant) relative risk aversion (or the inverse of the intertemporal elasticity of substitution). One can directly verify from (19) the following proposition:

Proposition 5 *Suppose that u exhibits constant relative risk aversion $\sigma > 0$, and that the production function is linear i.e., $f(x,\rho) = \rho x$ with $\underline{\rho} > 1$. Further, assume that $\beta E(\rho^{1-\sigma}) < 1$. Then, for any given $y > 0$, the (unique) optimal investment $x(y,\rho)$ is strictly increasing in ρ if $\sigma > 1$ (high risk aversion, low intertemporal substitution elasticity), strictly decreasing in ρ if $\sigma < 1$ and independent of ρ when $\sigma = 1$.*

Proposition 5 explains why one may not always expect investment to increase (and in fact, sometimes quite the reverse) when exogenous fluctuations cause increase in productivity, or return on investment, and this is anticipated by economic agents. Further, consumption preferences play a very important role here.

It is therefore of some interest to see if there are some more general conditions under which optimal investment increases with anticipation of a more productive shock. To examine the issues involved, we confine attention to production functions where the productivity shock is multiplicative i.e., assume:

(T.6) $f(x,\rho) = \rho h(x), \underline{\rho} = \inf A > 0$ and h satisfies all conditions needed to ensure **(T.1)-(T.5).**

Consider the functional equation of dynamic programming:

$$V(y,\rho) = \max_{0\leq x\leq y} u(y-x) + \beta E_{\rho'}[V(\rho h(x),\rho')] \tag{20}$$

Using **(T.5)** and the uniqueness and interiority of optimal policy, it can be shown that the value function is twice continuously differentiable and that the optimal policy function is continuously differentiable. Let

$$W(x,\rho) = E_{\rho'}V(\rho h(x),\rho') \tag{21}$$

Fix $y > 0$. Consider $\rho_1, \rho_2 \in A$ with $\rho_1 < \rho_2$, and let $x_1 = x(y,\rho_1)$ and $x_2 = x(y,\rho_2)$. Then, clearly $x_1, x_2 \in [0,y]$. If $x_1 \neq x_2$, then using (20) and 21:

$$u(y - x_1) + \beta W(x_1,\rho_1) \geq u(y - x_2) + \beta W(x_2,\rho_1)$$
$$u(y - x_2) + \beta W(x_2,\rho_2) \geq u(y - x_1) + \beta W(x_1,\rho_2)$$

so that

$$W(x_2,\rho_2) + W(x_1,\rho_1) \geq W(x_1,\rho_2) + W(x_2,\rho_1) \tag{22}$$

If the function $W(x,\rho)$ is **supermodular** on $\{(x,\rho) : 0 \leq x \leq y, \rho \in A\}$, then it is easy to show that $x_1 \leq x_2$. From (21), we have that $W_{x\rho} \geq 0$ and W is supermodular in (x,ρ) if the following inequality holds for all (x,ρ,ρ') :

$$-\frac{V_{11}(\rho h(x),\rho')}{V_1(\rho h(x),\rho')}\rho h(x) \leq 1$$

i.e.,

$$-\frac{V_{11}(y,\rho')y}{V_1(y,\rho')} \leq 1, \text{ for all } y > 0, \rho' \in A. \tag{23}$$

Thus, W is supermodular if the relative risk aversion exhibited by the *value function* is below 1, and in that case, optimal investment is weakly increasing in the shock. Note that as optimal policy is in the interior of the feasible set, using Theorem 1 of Edlin and Shannon (1998), one can check that if $W_{x\rho} > 0$, $x_1 < x_2$. Thus, optimal investment is strictly increasing in ρ if (23) holds strictly. If the inequality in (23) holds the other way, W is submodular in (x,ρ) and in that case, optimal investment is decreasing in ρ.[4] Thus, we have:

Lemma 6 *Assume (**T.5**) and (**T.6**). (i) Suppose that*

$$-\frac{V_{11}(y,\rho)y}{V_1(y,\rho)} \leq (<)1, \textit{ for all } y > 0, \rho \in A. \tag{24}$$

Then, for any $y > 0$, optimal investment $x(y,\rho)$ is (strictly) increasing in ρ.

(ii) Suppose that

$$-\frac{V_{11}(y,\rho)y}{V_1(y,\rho)} \geq (>)1, \textit{ for all } y > 0, \rho \in A. \tag{25}$$

Then, for any $y > 0$, optimal investment $x(y,\rho)$ is (strictly) decreasing in ρ..

[4] If $W(x,\rho)$ is **submodular** on $\{(x,\rho) : 0 \leq x \leq y, \rho \in A\}$, and $x_1 < x_2$. Then, from (22) we have:

$$W(x_1 \vee x_2, \rho_1 \vee \rho_2) + W(x_1 \wedge x_2, \rho_1 \wedge \rho_2) > W(x_2,\rho_1) + W(x_1,\rho_2)$$

which violates submodularity of W.

Lemma 6 indicates the role of relative risk aversion in determining the monotonicity of investment in productivity shock; however, the conditions in the lemma are in terms of the risk aversion displayed by the value function which is endogenous to the model. To be useful, we would like to have a condition in terms of the primitives of the model. It is, however, difficult in general to derive bounds on the risk aversion displayed by the value function through conditions on technology and preferences. In particular, the elasticities of both utility and production functions play a role in the curvature of the value function. The next proposition, which is one of the key contributions of the paper, provides one such characterization under the additional assumption that the production function exhibits bounded growth:

(T.7) $\lim_{x\to\infty} \frac{\overline{\rho}h(x)}{x} < 1$,where $\overline{\rho} = \sup A$.

For $\epsilon > 0$ small enough, define:

$$K = \inf\{x > 0 : \max_{\rho\in A} \rho h(x) \leq x\} + \epsilon. \tag{26}$$

$$\underline{\sigma} = \inf_{0<c<K}\{-\frac{u''(c)c}{u'(c)}\}, \overline{\sigma} = \sup_{0<c<K}\{-\frac{u''(c)c}{u'(c)}\} \tag{27}$$

Let $\eta(x)$ be the sum of first and second elasticity of the production function defined by:

$$\eta(x) = [\frac{h'(x)x}{h(x)} - \frac{h''(x)x}{h'(x)}], \quad x > 0.$$

Note that if $h(x) = x^\gamma, 0 < \gamma < 1$, then $\eta(x) = 1$, for all $x > 0$. Further, if $h(x) = \frac{Bx}{1+x}$ where $B > 1$, then $\eta(x) = \frac{1+2x}{1+x} > 1$, for all $x > 0$. Finally, if $h(x) = x^\alpha + x^\beta, 0 < \alpha < 1, 0 < \beta < 1$, $\alpha \neq \beta$, then $\eta(x) < 1$, for all $x > 0$.

Proposition 7 *Assume* (**T.5**), (**T.6**) *and* (**T.7**).

(a) Suppose that $\underline{\sigma} \geq 1$ *and* $\eta(x) \geq 1$ *for all* $x \in (0, K)$. *Then, for each* $y \in (0, K]$, *optimal investment* $x(y, \rho)$ *is non-increasing in* ρ *on* A.

(b) Suppose that $\overline{\sigma} \leq 1$ *and* $\eta(x) \leq 1$ *for all* $x \in (0, K)$. *Then, for each* $y \in (0, K]$, *optimal investment* $x(y, \rho)$ *is non-decreasing in* ρ *on* A.

Proposition 7 provides a set of verifiable sufficient condition on technology and preferences under which better shocks increase or decrease investment. From Lemma 6, we know that complementarity between investment and shocks depends on the curvature of the value function. The latter, in turn, is influenced by the curvature of the production and utility functions. For the specific case of linear production and constant relative risk aversion utility function, we have seen in Proposition 5 that investment is increasing or decreasing in the shock depending on whether relative risk aversion is above or below 1. Proposition 7 shows that for a more general class of production technology, even if relative risk aversion is not constant but uniformly bounded below by 1, investment is increasing in the shock as long as the sum of the first and second elasticity of the production function is bounded below by 1. Likewise, if relative risk aversion and the sum of the first and second elasticity of the production function are uniformly bounded above by 1, then investment decreases and consumption increases with a better shock. The degree of concavity of the utility and production functions are important determinants of how capital formation responds to forthcoming shocks.

4.2 Effect of Better Shocks on Output.

We now analyze how *output* changes with an improvement in the predicted realization of the shock next period. In the standard stochastic growth model where no information about the forthcoming shocks is available prior to consumption-investment decisions, investment depends only on current output. As a result, next period's output is always increasing (decreasing) in the realization of next period's shock as long as the production function is increasing (decreasing) in the shock. In our framework, next period's shock is known to the decision maker when she decides on consumption and investment and we have seen in the previous subsection, investment may be adjusted according to productivity shock and in particular, better productivity shock may reduce investment. Nonetheless, as we show next, the output resulting from investment (that is adjusted to the shock) increases with a better shock under fairly general circumstances.

Given the current shock ρ to the production function and the current stock y, we denote the output of the **next period** by $y'(y, \rho)$ where

$$y'(y, \rho) = f(x(y, \rho), \rho)$$

Proposition 8 *Assume* (**T.5**). *Then, $y'(y, \rho)$ is strictly increasing in ρ i.e., a better realization of the forthcoming productivity shock leads to higher output.*

The proof of this proposition is based on complementarity between output next period and the productivity shock. Under assumption (T.5), higher realization of the shock increases total and marginal productivity so that an increase in the anticipated realization of next period's shock, reduces the current marginal cost (in terms of consumption sacrifice) needed to attain any given level of output next period. Thus, while better shocks may increase or decrease investment, it increases aggregate output as long as total and marginal product are ordered by the realization of the shock.

It is important to emphasize that Proposition 8 requires that marginal productivity (and not just total product) increase with a better shock. We now provide an example where the total product is ordered by the shock but the marginal product is not; we show that the output next period is non-monotonic in the shock.

Example 9 *Consider a version of the example considered in Section 3.2 where*

$$u(c) = \ln c, f(x, \rho) = x^{-\rho}$$

where

$$-1 < \underline{\rho} = \inf A \leq \overline{\rho} = \sup A < 0.$$

Assume $y_0 \in (0, 1]$;this implies that consumption, investment and output paths lie in the interval $[0, 1]$ with probability one. Note that for any $x \in (0, 1)$, the total product $f(x, \rho)$ is strictly increasing in ρ on A. If we define the random variable

$$\widetilde{\rho} = -\rho,$$

then the production technology reduces to the exact form described in Section 3.2, and the optimal policy function is explicitly given by:

$$x(y, \widetilde{\rho}) = [\frac{\beta \widetilde{\rho}}{1 + \beta[\widetilde{\rho} - E(\widetilde{\rho})]}]y$$

and therefore,

$$y'(y,\widetilde{\rho}) = [\frac{\beta\widetilde{\rho}}{1+\beta[\widetilde{\rho}-E(\widetilde{\rho})]}]^{\widetilde{\rho}} y^{\widetilde{\rho}}$$

By straightforward calculations we obtain that:

$$\begin{aligned}\frac{d}{d\widetilde{\rho}}[\ln\ y'(y,\widetilde{\rho})] &= \ln y + \ln\widetilde{\rho} + 1 + \ln\beta - \ln[1+\beta(\widetilde{\rho}-E(\widetilde{\rho}))] - \frac{\beta\widetilde{\rho}}{1+\beta[\widetilde{\rho}-E(\widetilde{\rho})]} \\ &= 1 + \ln y + \ln\frac{\beta\widetilde{\rho}}{1+\beta(\widetilde{\rho}-E(\widetilde{\rho}))} - \frac{\beta\widetilde{\rho}}{1+\beta[\widetilde{\rho}-E(\widetilde{\rho})]}\end{aligned}$$

Then, for $0 < y < 1$,

$$\frac{d}{d\widetilde{\rho}}[\ln\ y'(y,\widetilde{\rho})] < 1 + \ln\frac{\beta\widetilde{\rho}}{1+\beta(\widetilde{\rho}-E(\widetilde{\rho}))} - \frac{\beta\widetilde{\rho}}{1+\beta[\widetilde{\rho}-E(\widetilde{\rho})]}$$

which is < 0 *for* $\widetilde{\rho}$ *close enough to zero. Choose the distribution of* ρ *such that* $E(\rho) = -\frac{1}{2}$, *i.e.,* $E(\widetilde{\rho}) = \frac{1}{2}$. *Observe that, as* $y \to 1, \widetilde{\rho} \to 1, \beta \to 1$

$$\begin{aligned}\frac{d}{d\widetilde{\rho}}[\ln\ y'(y,\widetilde{\rho})] &= 1 + \ln y + \ln\frac{\beta\widetilde{\rho}}{1+\beta(\widetilde{\rho}-E(\widetilde{\rho}))} - \frac{\beta\widetilde{\rho}}{1+\beta[\widetilde{\rho}-E(\widetilde{\rho})]} \\ &\to 1 + \ln\frac{1}{1+(1-E(\widetilde{\rho}))} - \frac{1}{1+[1-E(\widetilde{\rho})]} \\ &= \frac{1-E(\widetilde{\rho})}{1+[1-E(\widetilde{\rho})]} + \ln\frac{1}{1+(1-E(\widetilde{\rho}))} \\ &= \frac{1}{3} + \ln\frac{2}{3} > 0\end{aligned}$$

Therefore, there exists $\beta \in (0,1), y \in (0,1)$ *such that* $\frac{d}{d\widetilde{\rho}}[\ln\ y'(y,\widetilde{\rho})] > 0$ *for* $\widetilde{\rho}$ *close enough to 1. Fix any such* β, y *and choose* $\underline{\rho}$ *sufficiently close enough to* -1 *and* $\overline{\rho}$ *close enough to zero. Then, using the above arguments,* $y'(y,\rho)$ *is strictly increasing in* ρ *in a neighborhood of* $\overline{\rho}$ *and strictly decreasing in* ρ *in a neighborhood of* $\underline{\rho}$. *Thus, output next period is non-monotonic in* ρ.

5 Effect of Information about Shocks: Comparison with the Standard Model.

In this section, we discuss the effect of information about forthcoming productivity shocks and the ability to predict their realizations by comparing the dynamic optimal policy of our model to that in the standard stochastic growth model where no additional information about the realization of forthcoming productivity shocks is available to economic agents (before making their consumption-investment decisions). Much of the discussion in this section is based on some of the parametric family of utility and production functions discussed in Section 3. For ease of notation, we shall refer to the standard stochastic growth model with no additional information about the realization of the forthcoming shock as the NP-model, and to our model with short run prediction of forthcoming shock as the P-model. We denote the optimal investment function in the NP-model by $\widehat{x}(y)$, while, as before, we denote the optimal investment function in the P-model by $x(y,\rho)$.

5.1 Effect on Investment

First, consider the effect on investment. It is obvious that the comparison of $\widehat{x}(y)$ with $x(y,\rho)$ is likely to depend on the specific realization ρ of the forthcoming shock. To illustrate this clearly, consider the economy with the specific utility and production functions described in subsection 3.2 where $u(c) = \ln c$, $f(x,\rho) = x^\rho, 0 < \underline{\rho} = \inf A \leq \overline{\rho} = \sup A < 1$. As derived in that subsection, the optimal investment function in the P-model is then given by:

$$x(y,\rho) = [\frac{\beta\rho}{1+\beta[\rho - E(\rho)]}]y$$

In the NP-model with no information about the forthcoming shock, the optimal investment function has been derived in Mirman and Zilcha (1975), and is given by :

$$\widehat{x}(y) = \beta E(\rho)y$$

Observe that:

$$\rho \geq E(\rho) \Longleftrightarrow \frac{\beta\rho}{1+\beta[\rho - E(\rho)]} \geq \beta E(\rho)$$

so that $x(y,\rho) \geq \widehat{x}(y)$ if, and only if, $\rho \geq E(\rho)$. In other words, if the realization ρ of the forthcoming shock is above average, then investment is higher in the P model where the shock is predicted and the opposite is true when the realization is below average.

Next, consider the effect of information on the *ex ante* expected investment. Even here, the comparison can go either way depending on the parameters of the model. To illustrate this, we consider the specific case of a *linear production* technology and *CES utility* function described by (4), (5) and (6) with parametric restrictions (7) and (8). In our P-model with prior information about forthcoming shock, the optimal investment policy function is given by

$$x(y,\rho) = [\frac{(\mu\beta)^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}}}{1+(\mu\beta)^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}}}]y, \tag{28}$$

where σ is the constant elasticity of substitution and the constant $\mu > 1$ is implicitly determined by:

$$E[(\mu^{-\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}})^\sigma] = 1. \tag{29}$$

In the NP-model, the optimal investment function has been derived in the literature (see, for example, De Hek and Roy, 2001) and is given by

$$\widehat{x}(y) = [\beta E(\rho^{1-\sigma})]^{\frac{1}{\sigma}}y \tag{30}$$

Proposition 10 *Consider the economy with linear production function and CES utility function described by* $(4)-(8)$.

(a) If $\sigma > 1$, *then* $Ex(y,\rho) < \widehat{x}(y)$, *i.e. from any level of current output* $y > 0$, *expected current investment is* ***lower*** *in the P-model than in the NP-model.*

(b) Suppose that $\beta E(\rho) > 1$. *Then, there exists a range of admissible values of* $\sigma < 1$ *and* $\beta \in (0,1)$ *such that* $Ex(y,\rho) > \widehat{x}(y)$, *i.e. from any level of current output* $y > 0$, *expected investment is* ***higher*** *in the P-model than in the NP-model.*

Proposition 10 illustrates the important role played by consumption preferences in determining the qualitative effect of information about forthcoming shocks on capital formation. In particular, such information may increase or decrease (average) capital stocks depending on the curvature of the utility function or the degree of relative risk aversion.

5.2 Sensitivity of Output to Shock.

We now study the effect of information about forthcoming shock on the sensitivity of output to the random shock. In the standard stochastic growth model (the NP-model) where there is no additional information about the forthcoming shock, the output next period given current output y and for realization ρ of the random shock, is given by:

$$\widehat{y}(y,\rho) = f(\widehat{x}(y),\rho)$$

so that the sensitivity of $\widehat{y}$ to different realizations of ρ arises simply through the production function. In particular, suppose that assumption (T.5) holds. Then,

$$\frac{\partial \widehat{y}(y,\rho)}{\partial \rho} = \frac{\partial f(\widehat{x}(y),\rho)}{\partial \rho}. \tag{31}$$

On the other hand, in our P-model where the forthcoming shock is accurately predicted, next period's output is given by:

$$y'(y,\rho) = f(x(y,\rho),\rho)$$

so that under assumption (T.5),

$$\frac{\partial y'(y,\rho)}{\partial \rho} = \frac{\partial f(x(y,\rho),\rho)}{\partial x}\frac{\partial x(y,\rho)}{\partial \rho} + \frac{\partial f(x(y,\rho),\rho)}{\partial \rho}. \tag{32}$$

Now, because of differences in the investment levels $\widehat{x}(y)$ and $x(y,\rho)$, the right hand sides of (31) and (32) cannot be directly compared. However, if we confine attention to the case of multiplicative shock where:

$$f(x,\rho) = \rho h(x) \tag{33}$$

one can say something more specific, for in that case:

$$\frac{\partial \widehat{y}(y,\rho)}{\partial \rho} = h(\widehat{x}(y))$$

and therefore, the *elasticity of output* with respect to ρ in the NP-model is given by:

$$\eta_{\widehat{y},\rho} = \frac{\rho}{\widehat{y}}\frac{\partial \widehat{y}(y,\rho)}{\partial \rho} = \frac{\rho h(\widehat{x}(y))}{\widehat{y}} = 1 \tag{34}$$

whereas the elasticity of output with respect to ρ in the P-model is given by:

$$\eta_{y',\rho} = \frac{\rho}{y'}\frac{\partial y'(y,\rho)}{\partial \rho} = \frac{h'(x(y,\rho))}{h(x(y,\rho))}\frac{\partial x(y,\rho)}{\partial \rho}\rho + 1. \tag{35}$$

From (34) and (35),

$$\eta_{y',\rho} \geq \eta_{\widehat{y},\rho} \Longleftrightarrow \frac{\partial x(y,\rho)}{\partial \rho} \geq 0.$$

Thus, for a production function of the form (33), it follows that information about forthcoming shocks increases (decreases) the elasticity of output with respect to the production shock precisely in situations where the investment function $x(y, \rho)$ in the P-model is increasing (decreasing) in ρ; as we have seen in the previous section, the latter depends, among other things, on the extent of relative risk aversion and the elasticities of the production function.

5.3 Volatility of Output

We now compare the P and the NP models with respect to dispersion or volatility of output next period from any current level of output. This allows us to shed some light how information (about forthcoming shock) may affect the transmission of volatility of the shock to the output next period.Our analysis in this subsection will confine attention to the economy described by (4),(5) and (6) with parametric restrictions (7) and (8) where the production function is linear and the utility function is CES or constant relative risk aversion .

As discussed earlier, for this economy, in the P model with prior information about forthcoming shock, the optimal investment policy function is given by (28) and (29) so that the output next period is given by:

$$y'(y, \rho) = G(\rho)y$$

where

$$G(\rho) = [\frac{(\mu\beta)^{\frac{1}{\sigma}}\rho^{\frac{1}{\sigma}}}{1 + (\mu\beta)^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}}}]. \tag{36}$$

In the NP version of the model, the optimal investment function is given by (30)and the output next period is given by:

$$\widehat{y}(y, \rho) = [\rho\widehat{k}]y$$

where

$$\widehat{k} = [(E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}}]. \tag{37}$$

We would like to compare the volatility of

$$\frac{y'(y, \rho)}{y} = G(\rho) \tag{38}$$

with that of

$$\frac{\widehat{y}(y, \rho)}{y} = \rho\widehat{k}. \tag{39}$$

Let X and Y be two random variables and denote their zero-mean normalizations by $\widehat{X} = X - E(X)$ and $\widehat{Y} = Y - E(Y)$. We say that X is *more volatile or dispersed* than Y if the distribution of $\widehat{X}$ is a mean-preserving spread of the distribution of $\widehat{Y}$.[5]

Proposition 11 *Consider the economy with a linear production function and constant relative risk aversion (or, CES) utility function described by* (4), (5) *and* (6) *with parametric restrictions* (7) *and* (8)*. Let* $G(\rho)$ *be the function defined by* (36) *and* $\widehat{k}$, *the constant defined by* (37)*. Then the following hold:*

[5] This partial ordering of distributions with respect to dispersion is the Bickel-Lehman stochastic ordering (Landsberger and Meilijson, 1994).

(a) If $G'(\underline{\rho}) \leq \widehat{k}$ and the degree of relative risk aversion $\sigma > 1$, then for any given level of current output, output next period is more dispersed in the P-model than in the NP-model i.e., information about forthcoming shock increases the volatility of output.

(b) If $G'(\underline{\rho}) \geq \widehat{k}$ and the degree of relative risk aversion $\sigma < \frac{1}{2}$, then for any given level of current output, output next period is more dispersed in the NP-model than in the P- model i.e., information about forthcoming shock decreases the volatility of output.

The proposition indicates under certain verifiable conditions on the parameters, information about a forthcoming shock is likely to increase the volatility of output if relative risk aversion is large, and decrease the volatility of output if risk aversion is small. Once again this highlights the important role played by preferences in determining the effect of information on the nature of macroeconomic outcomes in the growth process.

6 Long Run Convergence

In this section, we discuss the long run behavior of the economy under short run prediction of the forthcoming shock. In particular, we confine attention to the case where the production technology exhibits bounded growth so that consumption, capital and output processes are uniformly bounded. For such a technology, it is well known that in the standard stochastic growth framework (NP-model), the optimal stochastic process of capital and output converge in distribution to a globally stable invariant distribution (under certain regularity conditions). In our model, where optimal investment in each period depends on both current output as well as the predicted realization of the forthcoming shock, it is by no means obvious that similar results should hold. We will show that under a set of assumptions that are comparable to ones imposed in the standard framework, and independent of initial economic conditions, optimal outputs converge in distribution to a unique invariant distribution whose support is bounded away from zero.

Given initial stock of output y_0 and the realization ρ_1 of the production shock in period 1 (observed in period 0), the stochastic process of optimal outputs $\{y_t\}_{t=0}^{\infty}$ is determined by the following law of motion:

$$y_{t+1} = f(x(y_t, \rho_{t+1}), \rho_{t+1}), t \geq 0.$$

Observe that given the optimal investment function $x(y, \rho)$ and the initial condition (y_0, ρ_1), $y_1 = f(x(y_0, \rho_1), \rho_1)$ is a deterministic number. We can therefore equivalently study the stochastic process of optimal outputs $\{y_t\}_{t=1}^{\infty}$ where the initial condition is y_1. Note that (using Lemma 1), $y_0 > 0$ implies that $y_1 > 0$ for all $\rho_1 \in A$, and $y_1 = 0$ for some $\rho_1 \in A$ if, and only if, $y_0 = 0$. Let:

$$H(y, \rho) = f(x(y, \rho), \rho).$$

$H(y, \rho)$ is the optimal transition function that relates current output to the optimal output next period for each realization of the random shock ρ. Since $f(z, \rho)$ is continuous and strictly increasing in z and $x(y, \rho)$ is continuous and strictly increasing in y (Lemma 2), it follows that $H(y, \rho)$ is continuous and strictly increasing in y on $\mathbb{R}_+$. Further, $H(0, \rho) = 0$ and for all $y > 0$, $H(y, \rho) > 0$ for all $\rho \in A$.

Given period 1 output $y_1 > 0$, the stochastic process of optimal output $\{y_t\}_{t=1}^{\infty}$ is given by

$$y_{t+1} = H(y_t, \rho_t), t \geq 1. \tag{40}$$

We will show that under certain conditions, for every $y_1 > 0$, the stationary Markov process $\{y_t\}_{t=1}^{\infty}$ as defined by (40) converges in distribution to a unique invariant distribution whose support is bounded away from zero.

We begin by imposing the following assumption:

(T.8) *Either A is finite or $f(x, \rho)$ is continuous in ρ on A.*

Let $\overline{f}(x), \underline{f}(x)$ be defined by:

$$\overline{f}(x) = \max_{\rho \in A} f(x, \rho), \underline{f}(x) = \min_{\rho \in A} f(x, \rho).$$

It is easy to check using (T.8), that $\overline{f}(x), \underline{f}(x)$ are continuous in x.

Next, we assume that the production function exhibits bounded growth:

(T.9) $\lim_{x \to \infty} \frac{\overline{f}(x)}{x} < 1$.

Let

$$K = \sup\{x : \overline{f}(x) \geq x\}$$

Under assumptions (T.4) and (T.9), $0 < K < \infty$.

Using the optimality equation (1) and the Maximum Theorem, one can show that if $f(x, \rho)$ is continuous in ρ on A, $x(y, \rho)$ and therefore $H(y, \rho) = f(x(y, \rho), \rho)$ is continuous in ρ on A. Let

$$\underline{H}(y) = \min_{\rho \in A} H(y, \rho), \overline{H}(y) = \max_{\rho \in A} H(y, \rho), y > 0.$$

Note that the minimum and the maximum above are well defined. Further, using the Maximum Theorem, $\underline{H}(y)$ and $\overline{H}(y)$ are continuous (and non-decreasing). $\underline{H}(y)$ and $\overline{H}(y)$ represent the worst and best optimal transition functions (the lowest and highest possible values of next period's output over all possible realizations of next period's shock, when current output is y).

By definition, $\overline{H}(y) \geq \underline{H}(y)$ for all y.We now impose a mild condition on the optimal transition functions:

(C.1) $\overline{H}(y) > \underline{H}(y)$ for all $y \in (0, K]$

Condition (C.1) ensures that under the optimal policy, the distribution of next period's output is non-degenerate. There are various conditions on technology and preferences that can ensure (C.1). For instance, if (T.5) holds, then from Proposition 8, $H(y, \rho)$ is strictly increasing in ρ so that (C.1) holds.

Next, we impose a condition on the "worst" optimal transition function:

(C.2) There exists $\alpha > 0$ such that $\underline{H}(y) > y, \forall y \in (0, \alpha)$.

Condition (C.2) requires that when current output is small enough, the optimal output next period is strictly higher than current output (i.e., the economy expands) even under the worst realization of the production shock. This ensures that independent of initial condition, long run output (and therefore the limiting distribution of output) is uniformly bounded away from zero. The next lemma provides verifiable sufficient conditions on preferences and technology under which (C.2) holds.

Let

$$\nu(x) = \inf_{\rho \in A} f'(x, \rho)$$

Lemma 12 *Suppose that at least one of the following holds:*

$$\lim_{x\to 0}\inf \nu(x)[\frac{u'(\overline{f}(x))}{u'(\underline{f}(x)-x)}] > \frac{1}{\beta} \tag{41}$$

$$A \text{ is finite, } \nu(x) \to +\infty \quad \text{as } x \to 0. \tag{42}$$

Then, (*C*.2) *holds i.e., there exists* $\alpha > 0$ *such that* $\underline{H}(y) > y, \forall y \in (0, \alpha)$.

The sufficient condition (42) for (C.2) is similar to conditions imposed in the standard stochastic growth literature (for instance, Brock and Mirman, 1972) to ensure that the economy is uniformly bounded away from zero almost surely in the long run. Sufficient condition (41) for (C.2) is similar to a condition used in a non-convex optimal stochastic growth model by Mitra and Roy (2006).

Example 13 *To see how* (41) *may be satisfied consider the case of the CES utility function given by* (4) *and* (5) *so that the marginal utility of consumption is given by:*

$$u'(c) = c^{-\sigma}, \sigma > 0.$$

Further, suppose that the random shock enters the production function multiplicatively:

$$f(x, \rho) = \rho h(x)$$

and $\underline{\rho} > 0$. *Then,* $\overline{f}(x) = \overline{\rho}h(x), \underline{f}(x) = \underline{\rho}h(x), \nu(x) = \underline{\rho}h'(x)$ *and (41) holds if:*

$$\underline{\rho}h'(0)[\frac{\underline{\rho}}{\overline{\rho}} - \frac{1}{\overline{\rho}h'(0)}]^{\sigma} > \frac{1}{\beta}.$$

This is satisfied for all $\beta \in (0, 1)$, *if* $h'(0) = +\infty$.

Define γ_0, γ_1 as follows:

$$\gamma_0 = \sup\{y > 0 : \underline{H}(y) \geq y\} \tag{43}$$

$$\gamma_1 = \inf\{y > 0 : \overline{H}(y) \leq y\} \tag{44}$$

Using condition (T.9), (C.1) and (C.2), it follows that γ_0 and γ_1 are well defined and:

$$0 < \gamma_0 \leq K, 0 < \gamma_1 \leq K.$$

γ_0 is the largest positive fixed point of the worst transition function and γ_1 is the smallest positive fixed point of the best transition function.

The next lemma lies at the heart of the uniqueness of invariant distribution; it ensures that every fixed point of the worst transition function $\underline{H}(y)$ lies below the smallest fixed point of the best transition function $\overline{H}(y)$.

Lemma 14 *Assume* (*T*.8)*,* (*T*.9)*,* (*C*.1) *and* (*C*.2)*. Then,* $\gamma_0 < \gamma_1$.

The rest of the steps leading to our main result follow similar arguments as in the existing literature on the standard stochastic growth model. Let ξ denote the probability measure for the random shock. For $t \geq 1$, define $\rho^t = (\rho_1, ..., \rho_t)$ and let ξ^t be the joint distribution of ρ^t. For each $n \geq 1$ and ρ^n, define $H^n(., \rho^n)$ by:

$$H^n(y_1, \rho^n) = H(.....H(H(y_1, \rho_2), \rho_3)....., \rho_n)$$

so that $H^n(y_1, \rho^n)$ is the realization of y_n given y_1 and $\rho^n = (\rho_2, ..., \rho_n)$. If μ is any probability on $\mathbb{R}_+$, define the probability $\xi^n\mu$ on $\mathbb{R}_+$ by

$$\xi^n\mu(B) = \int \xi^n(\{\rho^n : H^n(y_1, \rho^n) \in B) d\mu(y_1)$$

where B is any Borel subset of $\mathbb{R}_+$. $\xi^n\mu$ is the distribution of y_n when the distribution of y_1 is μ. μ is an invariant probability if $\xi^1\mu = \mu$. A subset S' of $\mathbb{R}_+$ is said to be $\xi-$invariant if it is closed and if

$$\xi(\{\rho \in A : H(y, \rho) \in S' \text{ for all } y \in S'\}) = 1.$$

A subset S'' of S' is a minimal $\xi-$invariant set if it is $\xi-$invariant and no strict subset of S'' is $\xi-$invariant. Finally, define y to be a $\xi-$fixed point if $\xi(\{\rho \in A : H(y, \rho) = y\}) = 1$. Following standard arguments used in stochastic growth models, we have:

Lemma 15 *Assume* $(T.8)$*,* $(T.9)$*,* $(C.1)$ *and* $(C.2)$*. For any* $c \in (0, \alpha)$*, the interval* $[c, K]$ *is* $\xi-$*invariant and* $[\gamma_{0,} \gamma_1]$ *is the unique minimal* $\xi-$*invariant interval in* $[c, K]$*.Further, there does not exist a* $\xi-$*fixed point in* $(0, K]$*.*

Given $y_1 > 0$, for $t > 1$, let $G_t(.)$ denote the probability distribution function of y_t. We are now ready to state the main result of this section.

Proposition 16 *Assume* $(T.8), (T.9), (C.1)$ *and* $(C.2)$*. Then, there is a unique invariant probability measure* μ *on* $\mathbb{R}_{++}$ *for the stochastic process* $\{y_t\}_{t=1}^{\infty}$ *and the support of this probability measure is the non-degenerate interval* $[\gamma_{0,} \gamma_1] \subset (0, K)$ *where* γ_0, γ_1 *are as defined in* (43) *and* (44)*. Further, independent of initial conditions,* $G_t(.)$*, the distribution function for the optimal output* y_t *in period* t*, converges uniformly as* $t \to \infty$ *to the distribution function for the probability measure* μ*.*

The proof of Proposition 16 follows directly from using Lemma 15 and showing that a "splitting condition" due to Dubins and Freedman (1966) is satisfied.

Though our qualitative result on convergence to a unique stochastic steady state (independent of initial condition) is similar to that obtained in the standard stochastic growth model (NP-model), the limiting steady states may differ significantly between the P and NP models. This is illustrated in the following example.

Example 17 *Consider the economy described in Section 3.2 where*

$$u(c) = \ln c, f(x, \rho) = \ x^{\rho}.$$

We will assume that at each date t, ρ_t can attain one of two possible values: $\underline{\rho} = 0.25$ *or* $\overline{\rho} = 0.75$ *with probability* $\frac{1}{2}$. *We have seen that in our model with short run prediction of forthcoming shock (P-model), the optimal investment policy function is given by:*

$$x(y,\rho) = [\frac{\beta\rho}{1+\beta[\rho - E(\rho)]}]y$$

so that

$$H(y,\rho) = [\frac{\beta\rho}{1+\beta[\rho - E(\rho)]}]^{\rho} y^{\rho}.$$

Choose $\beta = 0.5$. *Then, it is easy to check that* $H(y,\underline{\rho}) > H(y,\overline{\rho})$ *for all* $y \in (0,1)$. *Further, the function* $H(y,\rho)$ *has a unique positive fixed point. Setting* $\underline{H}(y) = H(y,\overline{\rho})$ *and* $\overline{H}(y) = H(y,\underline{\rho})$ *(and using (43), (44)), we have*

$$\gamma_0 = [\frac{\beta\overline{\rho}}{1+\beta[\overline{\rho} - E(\rho)]}]^{\frac{\overline{\rho}}{1-\overline{\rho}}} = (\frac{1}{7})^3, \gamma_1 = [\frac{\beta\underline{\rho}}{1+\beta[\underline{\rho} - E(\rho)]}]^{\frac{\underline{\rho}}{1-\underline{\rho}}} = (\frac{1}{3})^{\frac{1}{3}}.$$

As mentioned in the previous section, in the standard stochastic growth framework for this economy (NP-model), the optimal policy function is given by:

$$\widehat{x}(y) = \beta E(\rho) y$$

so that $\widehat{y}(y,\rho)$ *the optimal output next period when current output is* y *and the next shock has realization* ρ, *is given by:*

$$\widehat{y}(y,\rho) = f(\widehat{x}(y),\rho) = [\beta E(\rho) y]^{\rho}.$$

It is easy to check that given any $y_0 > 0$, *the stochastic process* $\{y_t\}_{t=0}^{\infty}$ *defined by* $y_{t+1} = \widehat{y}(y_t,\rho_{t+1})$ *converges to a unique invariant distribution whose support is the interval* $[m,M] \subset (0,1)$ *where* m *is the unique positive fixed point of the function* $[\beta E(\rho)y]^{\overline{\rho}}$ *and* M *is the unique positive fixed point of the function* $[\beta E(\rho)y]^{\underline{\rho}}$ *and, in particular,*

$$m = [\beta E(\rho)]^{\frac{\overline{\rho}}{1-\overline{\rho}}} = (\frac{1}{4})^3, M = [\beta E(\rho)]^{\frac{\underline{\rho}}{1-\underline{\rho}}} = (\frac{1}{4})^{\frac{1}{3}}.$$

Observe that

$$\gamma_0 < m < M < \gamma_1.$$

so that the support of the unique invariant distributions differ between the two models.

Thus, even though the difference in information structure of the two models (P and NP models) pertains only to the short run i.e., whether or not one can predict the immediately forthcoming shock, significant differences in the long run stochastic steady state of the economy may result.

APPENDIX.

Proof of Lemma 2.

Proof. The arguments used to prove these claims are similar to those used in the NP optimal growth model. However, for completeness, let us prove explicitly the strict monotonicity of $x(y,\rho)$ in y. Similarly, one can verify that of $c(y,\rho)$. Let $0 < y_1 < y_2$ and let $x_1 = x(y_1,\rho)$ and $x_2 = x(y_2,\rho)$. Assume to the contrary that $x_2 \leq x_1$.Since $x_2 \in [0, y_2]$ in this case, due to the uniqueness of the optimum, we can write that:

$$u(y_1 - x_1) + \beta E_{\rho'}\{V[f(x_1,\rho),\rho']\} \geq u(y_1 - x_2) + \beta E_{\rho'}\{V[f(x_2,\rho),\rho']\}$$

$$u(y_2 - x_1) + \beta E_{\rho'}\{V[f(x_1,\rho),\rho']\} \leq u(y_2 - x_2) + \beta E_{\rho'}\{V[f(x_2,\rho),\rho']\}$$

Since $V'(y_1,\rho) > V'(y_2,\rho)$ we must have $x_2 \neq x_1$, namely, $x_2 < x_1$, hence using the above two inequalities we obtain:

$$u(y_2 - x_1) - u(y_1 - x_1) < u(y_2 - x_2) - u(y_1 - x_2)$$

Denote: $y_2 = y_1 + \Delta$, where $\Delta > 0$, hence we attain that:

$$\frac{u(y_1 - x_1 + \Delta) - u(y_1 - x_1)}{\Delta} < \frac{u(y_1 - x_2 + \Delta) - u(y_1 - x_2)}{\Delta}$$

which is a contradiction due to the concavity of the utility function since $y_1 - x_1 < y_1 - x_2$. This proves that we must have $x_2 > x_1$. ■

Verification of Transversality Condition for the optimal policy in Section 3.1

Proof. Note that $y^*_{t+1} = \rho_{t+1}[1 - \lambda(\rho_t)]y^*_t$ and therefore,

$$y^*_{t+1} = \rho_{t+1}[1 - \lambda(\rho_{t+1})]y^*_t \leq [\prod_{j=1}^{t+1} \rho_j(1 - \lambda(\rho_j))]y^*_0, t = 0, 1, ...$$

which implies that

$$V'(y^*_t,\rho_t) = u'(c(y^*_t,\rho_t)) = [\lambda(\rho_t)y^*_t]^{-\sigma} = [\lambda(\rho_t)]^{-\sigma}[\prod_{j=1}^{t}(\rho_j)^{-\sigma}(1 - \lambda(\rho_j))^{-\sigma}]y_0$$

Since

$$1 - \lambda(\rho_j) = (\mu\beta)^{\frac{1}{\sigma}}\rho_j^{\frac{1-\sigma}{\sigma}} \frac{1}{1 + (\mu\beta)^{\frac{1}{\sigma}}\rho_j^{\frac{1-\sigma}{\sigma}}} = (\mu\beta)^{\frac{1}{\sigma}}\rho_j^{\frac{1-\sigma}{\sigma}}\lambda(\rho_j)$$

we have

$$(\rho_j)^{-\sigma}[1 - \lambda(\rho_j)]^{-\sigma} = (\mu\beta)^{-1}\rho_j^{-1}[\lambda(\rho_j)]^{-\sigma}$$

so that

$$\begin{aligned}
\beta^t EV'(y^*_t,\rho_t) &= \beta^t E[(\lambda(\rho_t))^{-\sigma}][\prod_{j=1}^{t} E[(\rho_j)^{-\sigma}(1 - \lambda(\rho_{j-1}))^{-\sigma}]y_0 \\
&= \beta^t\mu[(\mu\beta)^{-1}E\rho^{-1}(\lambda(\rho))^{-\sigma})]^{t-1} \\
&= \beta^{t-1}[E(\rho^{\frac{1}{\sigma}}\lambda(\rho))^{-\sigma})]^{t-1} < \beta^{t-1}[E(\lambda(\rho))^{-\sigma})]^{t-1}, \text{ as } \underline{\rho} > 1, \\
&= (\beta\mu)^{t-1} \to 0 \text{ as } t \to \infty
\end{aligned}$$

as $\beta\mu = \beta E(\rho^{-\sigma}) \leq \beta E(\rho^{1-\sigma}) < 1$ (using (8) and $\underline{\rho} > 1$). ■

Proof of Proposition 7

Proof. We will prove part (a). The proof of part (b) is essentially identical.
Set

$$V^0(y, \rho) = u(y), y \in [0, K], \rho \in A.$$

and for $t \geq 1$, define iteratively the functions $V^t(y, \rho)$ on $[0, K] \times A$ by

$$V^{t+1}(y, \rho) = \max_{0 \leq, x \leq y} \{u(y - x) + \delta E_{\rho'}[V^t(\rho h(x), \rho')]\}. \tag{45}$$

Note that V^t is the value function for a finite horizon version of the dynamic optimization problem (where there are t more periods left).

Step 1. We will show by induction that for all $t \geq 0$ and $\rho \in A, V^t(y, \rho)$ is continuous and concave in y on $[0, K]$, twice continuously differentiable in y,$V_1^t(y, \rho) > 0$ on $(0, K]$ and

$$-\frac{V_{11}^t(y, \rho)y}{V_1^t(y, \rho)} \leq 1, y \in (0, K], \rho \in A. \tag{46}$$

By assumption, this holds for $V^0(y, \rho) = u(y)$. Suppose that it holds for $t = T$. We will show that this holds for $t = T + 1$.Consider the functional equation (45) for $t = T$. Using strict concavity of u, strict concavity of h and concavity of $V^T(y, \rho)$ in y, it is easy to check that there is a unique solution $x^T(y, \rho)$ to the maximization problem on the right hand side of (45). Note that x^T is the optimal investment policy function for a finite horizon version of the dynamic optimization problem (where there are T more periods left). Further, using (U.3), $0 < x^T(y, \rho) < y$ for all $y \in (0, K], \rho \in A$. Using standard envelope arguments, one can then show that $V^{T+1}(y, \rho)$ is continuous and concave in y on $[0, K]$, twice continuously differentiable in y, $V_1^{T+1}(y, \rho) > 0$ and $x^T(y, \rho)$ is differentiable in y on $(0, K]$. Let $c^T(y, \rho) = y - x^T(y, \rho)$. Using the first order conditions for an interior solution to the maximization problem on the right hand side of (45) and the envelope theorem it follows that for all $\rho \in A, y \in (0, K]$:

$$V_1^{T+1}(y, \rho) = u'(c^T(y, \rho)) = \beta\rho h'(x^T(y, \rho))E_{\rho'}[V_1^T(\rho h(x^T(y, \rho)), \rho')].$$

and differentiating through this identity with respect to y we have:

$$\begin{aligned} V_{11}^{T+1}(y, \rho) = u''(c^T(y, \rho))c_1^T(y, \rho) = \beta x_1^T(y, \rho)[\rho h''(x^T(y, \rho))E_{\rho'}V_1^T(\rho h(x^T(y, \rho)), \rho') \\ +\{\rho h'(x^T(y, \rho))\}^2 E_{\rho'}\{V_{11}^T(\rho h(x^T(y, \rho)), \rho')\}] \end{aligned}$$

This implies that

$$\begin{aligned} -\frac{V_{11}^{T+1}(y, \rho)y}{V_1^{T+1}(y, \rho)} &= \{-\frac{u''(c^T(y, \rho))}{u'(c^T(y, \rho))}c^T(y, \rho)\}[\frac{c_1^T(y, \rho)y}{c^T(y, \rho)}] \\ &\geq \underline{\sigma}[\frac{c_1^T(y, \rho)y}{c^T(y, \rho)}] \end{aligned} \tag{47}$$

Further,

$$
\begin{aligned}
& -\frac{V_{11}^{T+1}(y,\rho)y}{V_1^{T+1}(y,\rho)} \\
= & -\frac{\beta x_1^T(y,\rho)y[\rho h''(x^T(y,\rho))E_{\rho'}V_1^T(\rho h(x^T(y,\rho)),\rho') + \{\rho h'(x^T(y,\rho))\}^2 E_{\rho'}\{V_{11}^T(\rho h(x^T(y,\rho)),\rho')\}]}{\beta\rho h'(x^T(y,\rho))E_{\rho'}[V_1^T(\rho h(x^T(y,\rho)),\rho')]} \\
= & \; x_1^T(y,\rho)y[\{-\frac{h''(x^T(y,\rho))}{h'(x^T(y,\rho))} \\
& +\frac{\rho h'(x^T(y,\rho))}{E_{\rho'}[V_1^T(\rho h(x^T(y,\rho)),\rho')]}E_{\rho'}\{-\frac{V_{11}^T(\rho h(x^T(y,\rho)),\rho')}{V_1^T(\rho h(x^T(y,\rho)),\rho')}\rho h(x^T(y,\rho))\frac{V_1^T(\rho h(x^T(y,\rho)),\rho')}{\rho h(x^T(y,\rho))}\}] \\
\geq & \; x_1^T(y,\rho)y[\{-\frac{h''(x^T(y,\rho))}{h'(x^T(y,\rho))} + \frac{\rho h'(x^T(y,\rho))}{E_{\rho'}[V_1^T(\rho h(x^T(y,\rho)),\rho')]}E_{\rho'}\{\underline{\sigma}\frac{V_1^T(\rho h(x^T(y,\rho)),\rho')}{\rho h(x^T(y,\rho))}\}] \\
= & \; \frac{x_1^T(y,\rho)y}{x^T(y,\rho)}[\eta(x^T(y,\rho)) + (\underline{\sigma}-1)\frac{h'(x^T(y,\rho))}{h(x^T(y,\rho))}] \\
\geq & \; \frac{x_1^T(y,\rho)y}{x^T(y,\rho)}. \qquad (48)
\end{aligned}
$$

where the last inequality follows from the conditions in the antecedent that $\underline{\sigma} \geq 1$ and $\eta(x) \geq 1$. It follows from (47) and (48) that

$$-\frac{V_{11}^{T+1}(y,\rho)y}{V_1^{T+1}(y,\rho)} \geq \max\{\frac{x_1^T(y,\rho)y}{x^T(y,\rho)}, \frac{c_1^T(y,\rho)y}{c^T(y,\rho)}\} \qquad (49)$$

There are only two possibilities:
(a) $\frac{c_1^T(y,\rho)y}{c^T(y,\rho)} \geq 1$
(b) $\frac{c_1^T(y,\rho)y}{c^T(y,\rho)} < 1$.
If (b) holds,:

$$c_1^T(y,\rho) \leq (\frac{c^T(y,\rho)}{y}) = 1 - \frac{x^T(y,\rho)}{y}$$

so that

$$x_1^T(y,\rho) = 1 - c_1^T(y,\rho) \geq \frac{x^T(y,\rho)}{y}$$

which implies:

$$\frac{x_1^T(y,\rho)y}{x^T(y,\rho)} \geq 1.$$

Thus:

$$\max\{\frac{x_1^T(y,\rho)y}{x^T(y,\rho)}, \frac{c_1^T(y,\rho)y}{c^T(y,\rho)}\} \geq 1 \qquad (50)$$

Using this in (49) implies that (46) holds for $t = T+1$. This completes Step 1.

Step 2. We now show that for all t, $x^t(y,\rho)$ is non-increasing in ρ (where $x^t(y,\rho)$ is the unique solution to the maximization problem on the right hand side of (45)). Let

$$W(x,\rho) = E_{\rho'}V^t(\rho h(x),\rho') \qquad (51)$$

Observe that for any given $\rho' \in A, V^t(\rho h(x), \rho')$ is twice continuously differentiation in (x, ρ) on $(0, K] \times A$ and

$$\frac{\partial^2 V^t(\rho h(x), \rho')}{\partial x \partial \rho} \leq 0$$

if

$$-\frac{V_{11}^t(\rho h(x), \rho')}{V_1^t(\rho h(x), \rho')}\rho h(x) \geq 1.$$

It follows from (46) in Step 1, therefore that

$$\frac{\partial^2}{\partial x \partial \rho} W(x, \rho) \leq 0 \tag{52}$$

on $\{(x, \rho) : 0 < x \leq y, \rho \in A\}$.Fix $y > 0$. Consider $\rho_1, \rho_2 \in A$ with $\rho_1 < \rho_2$, and let $x_1 = x^t(y, \rho_1)$ and $x_2 = x^t(y, \rho_2)$. We claim that $x_1 \geq x_2$.To see this, suppose to the contrary that $x_1 < x_2$. Clearly $x_1, x_2 \in (0, y)$. Using (45) and (51) and the uniqueness of solution to the maximization problem on the right hand side of (45):

$$u(y - x_1) + \beta W(x_1, \rho_1) > u(y - x_2) + \beta W(x_2, \rho_1)$$

$$u(y - x_2) + \beta W(x_2, \rho_2) > u(y - x_1) + \beta W(x_1, \rho_2)$$

so that

$$W(x_2, \rho_2) + W(x_1, \rho_1) > W(x_1, \rho_2) + W(x_2, \rho_1)$$

which violates (52).Thus, $x^t(y, \rho)$ is non-increasing in ρ for all t.

Step 3.For every $y \in (0, K]$, $x^t(y, \rho) \to x(y, \rho)$ as $t \to \infty$. This follows from Proposition 16.2 in Schäl (1975) that provides a condition under which optimal policy functions for finite horizon dynamic optimization problems converge to the optimal policy function for the infinite horizon problem as the horizon becomes infinitely large.

Finally, as $x^t(y, \rho)$ is non-increasing in ρ for every t, the (pointwise) limit $x(y, \rho)$ is non-decreasing in ρ. ■

Proof of Proposition 8

Proof. Let $h(y, \rho)$ be defined implicitly on $\mathbb{R}_+ \times A$ by:

$$f(h(y, \rho), \rho) = y \tag{53}$$

Thus, $h(y, \rho)$ is the investment required to attain output y next period when realization of the forthcoming productivity shock is ρ.It is easy to check that h is twice continuously differentiable on $\mathbb{R}_{++} \times A$ and that,

$$h_1 = \frac{1}{f_1} \tag{54}$$

$$h_2 = -\frac{f_2}{f_1} < 0 \tag{55}$$

and

$$\begin{aligned} h_{12} &= -\frac{1}{(f_1)^2}[f_{11}h_2 + f_{12}] \\ &< 0, \quad \text{since} \quad f_{12} > 0 \text{ (using (T.5))}. \end{aligned} \tag{56}$$

As ρ is observed prior to making investment decision, one can re-write the dynamic optimization problem as one where, given current output and realization ρ of next period's shock, the agent determines next period's output y'. The functional equation of dynamic programming can then be written as:

$$V(y, \rho) = \max_{0 \leq y' \leq f(y,\rho)} u(y - h(y', \rho)) + \beta E_{\rho'}[V(y', \rho'] \tag{57}$$

Fix $y > 0$. Consider $\rho_1 < \rho_2, \rho_1, \rho_2 \in A$ and let $y' = z_1$ be optimal from state (y, ρ_1) and $y' = z_2$ optimal from state (y, ρ_2).We first show that $z_1 \leq z_2$. Suppose, to the contrary, that $z_1 > z_2$. Since $z_1 \leq f(y, \rho_1), z_2 < f(y, \rho_1)$. Further, $z_1 \leq f(y, \rho_1) < f(y, \rho_2)$.From functional equation and the uniqueness of optimal actions:

$$\begin{aligned} u(y - h(z_1, \rho_1)) + \beta E_{\rho'}[V(z_1, \rho'] &> u(y - h(z_2, \rho_1)) + \beta E_{\rho'}[V(z_2, \rho'] \\ u(y - h(z_2, \rho_2)) + \beta E_{\rho'}[V(z_2, \rho'] &> u(y - h(z_1, \rho_2)) + \beta E_{\rho'}[V(z_1, \rho'] \end{aligned}$$

so that

$$u(y - h(z_1, \rho_1)) - u(y - h(z_2, \rho_1)) > u(y - h(z_1, \rho_2)) - u(y - h(z_2, \rho_2)) \tag{58}$$

Let

$$\phi(z, \rho) = u(y - h(z, \rho)).$$

Note that

$$\phi_1 = -u'(y - h(z, \rho))h_1$$

and

$$\phi_{12} = -u'(y - h(z, \rho))h_{12} + u''(y - h(z, \rho))h_1 h_2 > 0$$

From (58)

$$\phi(z_1, \rho_1) + \phi(z_2, \rho_2) \geq \phi(z_1, \rho_2) + \phi(z_2, \rho_1)$$

which leads to a contradiction as $\phi_{12} > 0$. Next, we claim that, in fact, $z_1 < z_2$. To see this, suppose to the contrary that

$$z_1 = z_2 = z.$$

Then (under assumption of uniqueness of optimal actions) since $\rho_1 < \rho_2$,

$$x(y, \rho_1) = x_1 > x(y, \rho_2) = x_2$$

where

$$f(x_1, \rho_1) = f(x_2, \rho_2) = z.$$

From the Ramsey-Euler equation (3), we have:

$$\begin{aligned} u'(y - x_1) &= \beta f_1(x_1, \rho_1) E_{\rho'}[u'(z - x(z, \rho'))] \\ u'(y - x_2) &= \beta f_1(x_2, \rho_2) E_{\rho'}[u'(z - x(z, \rho'))] \end{aligned}$$

so that

$$\frac{u'(y - x_1)}{u'(y - x_2)} = \frac{f_1(x_1, \rho_1)}{f_1(x_2, \rho_2)}$$

Observe that since $f_{11} < 0$, $f_{12} \geq 0, x_1 > x_2, \rho_1 < \rho_2$,

$$f_1(x_1, \rho_1) < f_1(x_2, \rho_1) \leq f_1(x_2, \rho_2)$$

while using strict concavity of u,

$$u'(y - x_1) > u'(y - x_2)$$

leading to a contradiction. This completes the proof. ■

Proof of Proposition 10

Proof. (a) Rewriting the expression for $x(y, \rho)$ we obtain that:

$$Ex(y, \rho) = [1 - E\frac{1}{1 + \mu^{\frac{1}{\sigma}}(\beta\rho^{1-\sigma})^{\frac{1}{\sigma}}}]\ y \tag{59}$$

Let $A = (\beta\mu)^{\frac{1}{\sigma}}$, $G(\rho) = \frac{A}{A+\rho^{\frac{1}{\sigma}}}$. Differentiating $G(\rho)$ twice we obtain that $sign\{G''(\rho)\} = sign\{-m(m-1)(\rho^m + A) + 2m^2\rho^m\} > 0$ for $m = \frac{1}{\sigma}$ and $\sigma \geq 1$. Let $z = \beta\mu\rho^{1-\sigma}$, then

$$\frac{1}{1 + \mu^{\frac{1}{\sigma}}(\beta\rho^{1-\sigma})^{\frac{1}{\sigma}}} = \frac{1}{1 + (z)^{\frac{1}{\sigma}}} = G(z)$$

which is strictly convex in z so that using Jensen's inequality:

$$EG(z) = E\frac{1}{1 + (z)^{\frac{1}{\sigma}}} > G(Ez) = \frac{1}{1 + (Ez)^{\frac{1}{\sigma}}}$$

and using this in (59)

$$\frac{Ex(y, \rho)}{y} < 1 - \frac{1}{1 + \mu^{\frac{1}{\sigma}}(E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}}} = \frac{(E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}}}{\mu^{-\frac{1}{\sigma}} + (E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}}} \tag{60}$$

Now, we show that for $\sigma \geq 1$ we have :

$$\mu^{-\frac{1}{\sigma}} + (E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}} \geq 1. \tag{61}$$

Define, $H(z) = [B + z^{\frac{1}{\sigma}}]^{\sigma}$ where $B = \mu^{-\frac{1}{\sigma}}, z = \beta\rho^{1-\sigma}$. Then,

$$signH''(z) = sign\{\frac{\sigma - 1}{\sigma}z^{\frac{1}{\sigma}} + (\frac{1}{\sigma} - 1)(B + z^{\frac{1}{\sigma}})\} \leq 0$$

for $\sigma \geq 1$. Therefore, using Jensen's inequality:

$$\begin{aligned} \mu^{-\frac{1}{\sigma}} + (E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}} &= H(E(z)) \geq EH(z) \\ &= E[(\mu^{-\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}})^{\sigma}] = 1, \end{aligned}$$

using (14). This establishes (61). Using (61) in (60) we obtain:

$$\frac{Ex(y, \rho)}{y} < (E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}} = \frac{\widehat{x}(y)}{y}$$

and this establishes the first part of the proposition.

(b) Consider $\sigma < 1$. For low values of σ, the condition (8) may not hold. However, it is easy to check that since $\beta E\rho > 1$, there exists $0 < \widehat{\sigma} < 1$ such that $(E\beta\rho^{1-\widehat{\sigma}})^{\frac{1}{\widehat{\sigma}}} = 1$ and therefore, (8) holds for $\sigma > \widehat{\sigma}$. Further, there exists $h \in (0,1)$, such that $\widehat{\sigma} < \frac{1}{2}$ for $\beta \in (0,h)$. Define,

$$L(z) = [1 + Dz^{\frac{1}{\sigma}}]^{-1} \qquad ; \text{where } \; D = \mu^{\frac{1}{\sigma}} \quad \text{and} \quad z = \beta\rho^{1-\sigma}$$

Differentiating this function twice we obtain that:

$$sign\{L''(z)\} = sign\{1 - \frac{1}{\sigma} + \frac{2}{\sigma}[\frac{Dz^{\frac{1}{\sigma}}}{1 + Dz^{\frac{1}{\sigma}}}]\}$$

Consider $\beta \in (0,h)$ so that $\widehat{\sigma} < \frac{1}{2}$ and consider $\sigma \in (\widehat{\sigma}, \frac{1}{2})$. Using (14), we have that $\mu \longrightarrow 1$ as $\beta \longrightarrow 0$. Further, $z = \beta\rho^{1-\sigma} \rightarrow 0$ as $\beta \rightarrow 0$. Thus, by choosing β small we can guarantee that $\frac{Dz^{\frac{1}{\sigma}}}{1+Dz^{\frac{1}{\sigma}}}$ is sufficiently small for all ρ, so that $L''(z) < 0$.Using the strict concavity of $L(z)$ we attain:

$$\begin{aligned}
\frac{Ex(y,\rho)}{y} &= E[1 - \frac{1}{1 + \mu^{\frac{1}{\sigma}}(\beta\rho^{1-\sigma})^{\frac{1}{\sigma}}}] = E[1 - L(z)] \\
&> [1 - L(E(z))] = 1 - \frac{1}{1 + \mu^{\frac{1}{\sigma}}(E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}}} \\
&= \frac{(E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}}}{\mu^{-\frac{1}{\sigma}} + (E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}}} \qquad (62)
\end{aligned}$$

Now, we show that for $\sigma < 1$ we have :

$$\mu^{-\frac{1}{\sigma}} + (E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}} < 1. \qquad (63)$$

Define, $H(z) = [B + z^{\frac{1}{\sigma}}]^{\sigma}$ where $B = \mu^{-\frac{1}{\sigma}}, z = \beta\rho^{1-\sigma}$. Then,

$$\begin{aligned}
signH''(z) &= sign\{\frac{\sigma - 1}{\sigma}z^{\frac{1}{\sigma}} + (\frac{1}{\sigma} - 1)(B + z^{\frac{1}{\sigma}})\} \\
&= sign\{(\frac{1}{\sigma} - 1)B\} > 0
\end{aligned}$$

for $\sigma < 1$. Therefore, using Jensen's inequality:

$$\begin{aligned}
\mu^{-\frac{1}{\sigma}} + (E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}} &= H(E(z)) < EH(z) \\
&= E[(\mu^{-\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}}\rho^{\frac{1-\sigma}{\sigma}})^{\sigma}] = 1,
\end{aligned}$$

using (14). This establishes (63). Using (62) and (63), we have

$$\frac{Ex(y,\rho)}{y} > (E\beta\rho^{1-\sigma})^{\frac{1}{\sigma}} = \frac{\widehat{x}(y)}{y}.$$

This completes the proof. ■

Proof of Proposition 11

Proof. We begin by observing that for $\sigma > 1$, $G(\rho)$, as defined in (36), satisfies $G' > 0, G'' < 0$ so that G is strictly increasing and strictly concave on $[\underline{\rho}\ , \overline{\rho}]$. We want to show that if $G'(\underline{\rho}) \leq \widehat{k}$, then $\widehat{k}\rho - E[\widehat{k}\rho]$ is a mean-preserving spread of $G(\rho) - E[G(\rho)]$. For this, it is enough to show that each expected utility maximizing risk averse decision maker will prefer the random variable $G(\rho) - E[G(\rho)]$ than $\widehat{k}\rho - E[\widehat{k}\rho]$ (see, Rothschild and Stiglitz,1970). Let U be any strictly concave and non-decreasing utility function on $\mathbb{R}$; without loss of generality, let U be differentiable. Then,

$$\begin{aligned} & E\{U(G(\rho) - E[G(\rho)]) - U(\widehat{k}\rho - E[\widehat{k}\rho]\} \\ \geq\ & E\{U'(G(\rho) - E[G(\rho)])[G(\rho) - E[G(\rho)] - [\widehat{k}\rho - E[\widehat{k}\rho]]\} \\ =\ & Cov\{U'(G(\rho) - E[G(\rho)]), G(\rho) - \widehat{k}\rho\} \geq 0 \end{aligned}$$

where the non-negativity of the covariance follows from the fact that U' is decreasing, $G' > 0$ and $G'(\rho) - \widehat{k} \leq G'(\underline{\rho}) - \widehat{k} \leq 0$ which together imply that $U'(G(\rho) - E[G(\rho)])$ is decreasing in ρ while $G(\rho) - \widehat{k}\rho$ is weakly decreasing in ρ. Thus, $y'(y, \rho) = G(\rho)y$ is more dispersed than $\widehat{y}(y, \rho) = \widehat{k}\rho y$. This completes proof of part (a). Next, we prove part (b). We can verify that for $\sigma < \frac{1}{2}$ we have $G'' > 0$ so that G is strictly increasing and strictly convex on $[\underline{\rho}\ , \overline{\rho}]$. We want to show that if $G'(\underline{\rho}) \geq \widehat{k}$, then $G(\rho) - E[G(\rho)]$ is a mean-preserving spread of $\widehat{k}\rho - E[\widehat{k}\rho]$.As before, let U be any strictly concave and non-decreasing utility function on $\mathbb{R}$; without loss of generality, let U be differentiable. Then,

$$\begin{aligned} & E\{U(G(\rho) - E[G(\rho)]) - U(\widehat{k}\rho - E[\widehat{k}\rho]\} \\ \leq\ & E\{U'(\widehat{k}\rho - E[\widehat{k}\rho])[G(\rho) - E[G(\rho)] - [\widehat{k}\rho - E[\widehat{k}\rho]]\} \\ =\ & Cov\{U'(\widehat{k}\rho - E[\widehat{k}\rho]), G(\rho) - \widehat{k}\rho - E[G(\rho)] - E[\widehat{k}\rho]\} \\ \leq\ & 0 \end{aligned}$$

where the negativity of the covariance follows from the fact that U' is decreasing and $G'(\rho) - \widehat{k} \geq G'(\underline{\rho}) - \widehat{k} \geq 0$ which together imply that $U'(\widehat{k}\rho - E[\widehat{k}\rho])$ is decreasing in ρ while $G(\rho) - \widehat{k}\rho$ is weakly increasing in ρ. Thus, $\widehat{y}(y, \rho) = \widehat{k}\rho y$ is more dispersed than $y'(y, \rho) = G(\rho)y$. This completes proof of part (b). ■

Proof of Lemma 12.

Proof. Suppose that, contrary to the lemma, there exists a strictly positive sequence $\{y_n\}_{n=1}^{\infty} \to 0$ such that

$$\underline{H}(y_n) \leq y_n, \forall n. \tag{64}$$

Let $\{x_n\}, \{\rho_n\}$ be defined by

$$x_n = x(y_n, \rho_n), \underline{H}(y_n) = f(x_n, \rho_n).$$

Since, $x_n \leq y_n, \{x_n\} \to 0$. From the Ramsey-Euler equation:

$$\begin{aligned} u'(c(y_n, \rho_n)) &= \beta f'(x(y_n, \rho_n), \rho_n) E_{\rho'}\{u'(c(f(x(y_n, \rho_n), \rho_n), \rho'))\} \\ &= \beta f'(x_n, \rho_n) E_{\rho'}\{u'(c(f(x_n, \rho_n), \rho'))\} \end{aligned} \tag{65}$$

First, suppose that (41) holds. Then, from (65)

$$u'(c(y_n,\rho_n)) \geq \beta f'(x_n,\rho_n)u'(f(x_n,\rho_n)), \text{ since } c(f(x_n,\rho_n),\rho') \leq f(x_n,\rho_n), \forall \rho'$$

and since $c(y_n,\rho_n) = y_n - x_n \geq f(x_n,\rho_n) - x_n$, we have

$$u'(f(x_n,\rho_n) - x_n) \geq \beta f'(x_n,\rho_n)u'(f(x_n,\rho_n))$$

and therefore,$\forall n$

$$1 \geq \beta f'(x_n,\rho_n)\frac{u'(f(x_n,\rho_n))}{u'(f(x_n,\rho_n)-x_n)} \geq \beta\nu(x_n)\frac{u'(\overline{f}(x_n))}{u'(\underline{f}(x_n)-x_n)},$$

which contradicts (41). Next, suppose that (42) holds. From (65):

$$\begin{aligned}
u'(c(y_n,\rho_n)) &= \beta f'(x_n,\rho_n)E_{\rho'}\{u'(c(f(x_n,\rho_n),\rho'))\} \\
&= \beta f'(x_n,\rho_n)E_{\rho'}\{u'(c(\underline{H}(y_n),\rho'))\} \\
&\geq \beta\nu(x_n)E_{\rho'}\{u'(c(\underline{H}(y_n),\rho'))\} \\
&\geq \beta\nu(x_n)E_{\rho'}\{u'(c(y_n,\rho'))\}, \text{ using (64)} \\
&\geq \beta\nu(x_n)u'(c(y_n,\rho_n))\Pr\{\rho'=\rho_n\} \\
&\geq \beta\nu(x_n)u'(c(y_n,\rho_n))q \text{ where } q = \min_{r\in A}\Pr\{\rho'=r\}
\end{aligned}$$

and as $q > 0$,we have

$$\nu(x_n) \leq \frac{1}{\beta q}, \forall n$$

which contradicts (42). ∎

Proof of Lemma 14

Proof. Suppose not. Using (C.1), $\gamma_0 \neq \gamma_1$. Therefore,

$$\gamma_0 > \gamma_1.$$

Since $\underline{H}(y)$ and $\overline{H}(y)$ are continuous, using (C.2), $\gamma_0 > \gamma_1 > \alpha > 0$ so that

$$\underline{H}(\gamma_0) = \gamma_0, \overline{H}(\gamma_1) = \gamma_1.$$

This implies that for all $\rho \in A$,

$$f(x(\gamma_0,\rho),\rho) \geq \underline{H}(\gamma_0) = \gamma_0. \tag{66}$$

$$f(x(\gamma_1,\rho),\rho) \leq \overline{H}(\gamma_1) = \gamma_1. \tag{67}$$

From (3):

$$\begin{aligned}
u'(c(\gamma_0,\rho)) &= \beta f'(x(\gamma_0,\rho),\rho)E_{\rho'}\{u'(c(f(x(\gamma_0,\rho),\rho),\rho'))\} \\
&\leq \beta f'(x(\gamma_0,\rho),\rho)E_{\rho'}\{u'(c(\gamma_0,\rho'))\}, \forall \rho \in A \text{ (using (66))}
\end{aligned}$$

so that by taking expectation with respect to ρ on both sides of the above inequality we have:

$$E_\rho[u'(c(\gamma_0,\rho))] \leq \beta E_\rho[f'(x(\gamma_0,\rho),\rho)]E_{\rho'}\{u'(c(\gamma_0,\rho'))\}$$

and noting that ρ, ρ' are iid random variables we have:

$$\beta E_\rho[f'(x(\gamma_0, \rho), \rho)] \geq 1$$

and since $\gamma_0 > \gamma_1$, strict concavity of $f(x, \rho)$ in x and the fact that $x(y, \rho)$ is strictly increasing in y implies that:

$$\beta E_\rho[f'(x(\gamma_1, \rho), \rho)] > 1. \tag{68}$$

Once again from (3):

$$\begin{aligned} u'(c(\gamma_1, \rho)) &= \beta f'(x(\gamma_1, \rho), \rho) E_{\rho'}\{u'[(c(f(x(\gamma_1, \rho), \rho), \rho'))\} \\ &\geq \beta f'(x(\gamma_1, \rho), \rho) E_{\rho'}\{u'(c(\gamma_1, \rho'))\}, \forall \rho \in A \text{ (using (67))}, \end{aligned}$$

so that by taking expectation with respect to ρ on both sides of the above inequality we have:

$$E_\rho[u'(c(\gamma_1, \rho))] \geq \beta E_\rho[f'(x(\gamma_1, \rho), \rho)] E_{\rho'}\{u'(c(\gamma_1, \rho'))\}$$

and noting that ρ, ρ' are iid random variables we have:

$$\beta E_\rho[f'(x(\gamma_1, \rho), \rho)] \leq 1$$

which contradicts (68). ■

Proof of Lemma 15

Proof. For any $y \in [c, K]$, $H(y, \rho) = f(x(y, \rho), \rho) \leq f(y, \rho) \leq K$ with probability one and further $H(y, \rho) = f(x(y, \rho), \rho) \geq f(x(c, \rho), \rho) = H(c, \rho) \geq \underline{H}(c) > c$, with probability one. Thus, $[c, K]$ is $\xi-$invariant. From Lemma 14, $[\gamma_{0,}\gamma_1]$ is a closed sub of $[c, K]$ for any $c \in (0, \alpha)$. Further, for any $y \in [\gamma_{0,}\gamma_1]$, $H(y, \rho) \leq H(\gamma_1, \rho) \leq \overline{H}(\gamma_1) = \gamma_1$ with probability one and further, $H(y, \rho) \geq H(\gamma_0, \rho) \geq \underline{H}(\gamma_0) = \gamma_0$ with probability one. Thus, $[\gamma_0, \gamma_1]$ is $\xi-$invariant.

We now show that is no $\xi-$invariant closed interval that is a strict subset of $[\gamma_0, \gamma_1]$. Suppose not. Then there exists a $\xi-$invariant closed interval $[s, r] \subsetneq [\gamma_0, \gamma_1]$. Then, either $s > \gamma_0$ or $r < \gamma_1$ or both. If $s > \gamma_0$, then $\underline{H}(s) < s$. This implies there exists $\rho(s) \in A$ such that $f(x(s, \rho(s)), \rho(s)) < s$.If A is finite, then $\xi\{\rho = \rho(s)\} > 0$ which immediately contradicts $\xi-$invariance of $[s, r]$. If A is not finite, then using (T.8) there exists $\delta > 0$, such that $H(s, \rho) < s$, for all $\rho \in A \cap (\rho(s) - \delta, \rho(s) + \delta)$ and since $\rho(s) \in A$, $\xi\{\rho : \rho \in A \cap (\rho(s) - \delta, \rho(s) + \delta)\} > 0$. This contradicts $\xi-$invariance of $[s, r]$. If $r < \gamma_1$, then $\overline{H}(r) > r$. This implies there exists $\rho(r) \in A$ such that $f(x(r, \rho(r)), \rho(r)) > r$. If A is finite, then $\xi\{\rho = \rho(r)\} > 0$ which immediately contradicts $\xi-$invariance of $[s, r]$. If A is not finite, then using (T.8), there exists $\delta > 0$, such that $H(r, \rho) > r$, for all $\rho \in A \cap (\rho(r) - \delta, \rho(r) + \delta)$ and since $\xi\{\rho : \rho \in A \cap (\rho(r) - \delta, \rho(r) + \delta)\} > 0$, we have a contradiction to the $\xi-$invariance of $[s, r]$.

Next, we argue that there is no other closed sub-interval of $[c, K]$ that is minimal $\xi-$invariant. To see this, suppose there is such an interval $[s, r] \neq [\gamma_0, \gamma_1]$. If $r < \gamma_1$, then $\overline{H}(r) > r$ and by the same argument as at the end of the last paragraph, we obtain a contradiction. Therefore, $r \geq \gamma_1$. As $[\gamma_0, \gamma_1]$ is a minimal $\xi-$invariant interval, $[s, r]$ is not a subset of $[\gamma_0, \gamma_1]$. As $[s, r]$ is a minimal $\xi-$invariant interval, $[\gamma_0, \gamma_1]$ is not a subset of $[s, r]$. Together these imply that $\gamma_0 < s$ that, in turn, implies that $\underline{H}(s) < s$. Using the same argument as in previous paragraph, we obtain a contradiction.

Finally, we observe that as $\gamma_0 < \gamma_1, \overline{H}(y) > y$ for all $y \in (0, \gamma_1)$ so that (using similar argument as above), $\xi(\{\rho \in A : H(y, \rho) > y\}) > 0$. Similarly, as $\gamma_0 < \gamma_1$,for all $y \geq \gamma_1 > \gamma_0$,

$\underline{H}(y) < y$ so that $\xi(\{\rho \in A : H(y, \rho) < y\}) > 0$. Thus, there there does not exist a $\xi-$fixed point in $(0, K]$ ■

Proof of Proposition 16

Proof. The proof is based on an appeal to results originally contained in Dubins and Freedman (1966, Corollary 5.5)[6] and adapted by Majumdar, Mitra and Nyarko (1989). In particular, we use Theorem 10 in Majumdar, Mitra and Nyarko (1989) that can be reported as follows (using our notation):

Let S'be a $\xi-$invariant closed interval in $[0, K]$. Suppose that for $\xi-$a.e. ρ in A, $H(., \rho)$ is continuous and non-decreasing on S' and there are no $\xi-$fixed points in A.If there is a unique minimal $\xi-$invariant closed interval in S' then for some integer n, ξ^n splits and the conclusions of Theorem 9 hold i.e., there is one and only one invariant probability μ on S' and for each probability $\widehat{\mu}$ whose support is a subset of S', the distribution function of $\xi^n\widehat{\mu}$ converges uniformly to the distribution function of μ.

Choosing $S' = [c, K]$ for any $c \in (0, \alpha)$ and $[\gamma_{0,}\gamma_1]$ as the candidate *unique minimal $\xi-$invariant closed interval in S'*, the proposition follows from Lemma 15. ■

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