A Simple Model of Competition Between Teams*

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Abstract

We model a competition between two teams (that may differ in size) as an all-pay auction with incomplete information. Individuals exert effort to increase the performance of one's own team via an additively separable aggregation function. The team with a higher performance wins, and its members enjoy the prize as a public good. The value of the prize is identical to members of the same team but is unknown to the other team. We show that in any monotone equilibrium in which everyone actively participates, the bigger team is more likely to win if the aggregation function is concave, less likely if convex, or equally likely if linear. We also show that if the aggregation function is concave or linear then the expected payoff for a player in the bigger team is higher than that in the smaller team. Finally, we investigate how teams may form endogenously.

1 Introduction

Many economic, political and social activities are performed by groups or organizations rather than individuals. When firms compete, the strategic interaction is really between *collectives* of individuals that make up the firms. Electoral competition between candidates involves strategic interaction between *teams* consisting of the candidates themselves, their consultants and the activists that support them. Lobbying efforts are carried out by interest *groups* who need to coordinate the actions of their members in response to the actions of other interest groups. Likewise, ethnic conflicts involve different *peoples* who are united by a common background such as religion, origin or economic status.

Despite the ubiquity of strategic interactions between groups, the majority of economic analysis treats players in game theoretic models as individual entities.

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While this may be a helpful simplification, it ignores the interplay between *intra*-group strategizing (how each member of a group reasons about the actions of other members of the *same* group) and *inter*-group strategizing (how each member of a group reasons about the actions of the members of the *opposing* groups), which may have important implications for the outcome of the interaction. Furthermore, by explicitly modeling each participating unit as a collective of decision-makers, one may be able to gain insights on how different group attributes (such as size, for instance) can affect the outcome.

In light of this we propose and analyze a model of competition between groups. We focus on the case of two competing groups, or teams, of possibly different sizes. Following the literature on group contests (see below) we model the interaction between teams as a generalized all-pay auction with incomplete information. Individual members of each team exert effort to increase the performance of one's own team via an additively separable aggregation function, and the team with a higher performance wins. Each player individually bears the cost of his own effort regardless of winning or not.

Since we assume that individuals act non-cooperatively - each maximizing his own personal payoff - a natural modeling question that arises is what makes a collection of individuals a "team"? We propose to view a team as a cohesive set of individuals who share the same values, which are commonly known to them. We therefore assume that when a team wins, its members enjoy the prize as a public good, and the value of this prize is identical to members of the same team but is unknown to the other team.

We consider aggregation functions that are either concave or convex, which we interpret as capturing either a case of physical tasks that typically exhibit decreasing returns to effort, or a case of "cognitive" tasks such as research or innovation where returns to effort may be increasing. We show that if the aggregation function is strictly concave or convex then there is a unique monotone equilibrium in which every player actively participates, that is, everyone puts positive effort in expectation. If the aggregation function is linear, then all monotone equilibria are equivalent at the team level. Subsequent analysis focuses on equilibria with active participation.

Our main interest is in understanding the implication of team size on the probability of winning and on the members' payoffs. This relates to the well-known "group size paradox", which argues that free-rding puts a bigger group at a disadvantage (see Olson (1982) and Becker (1983)). The main finding is that these implications are determined by the curvature of the aggregation function. We first show that the bigger team is more likely to win if the aggregation function is concave, less likely if convex, or equally likely if linear. The underlying intuition is that the curvature of the aggregation function determines whether *in equilibrium* additional members effectively augment or reduce the productivity of existent members. Second, we show that when the aggregation function is concave or linear, then the expected payoff for a player in the bigger team is higher than the expected payoff for for one in the smaller team. Moreover, there also exist convex aggregation functions under which the same result holds, despite the fact that the bigger team is less likely to win.

Since group size can have important implications, we investigate how teams might form. We consider a two stage game such that in the first stage players split into teams, and in the second stage the teams compete. Our main result is that team formation depends on how each member's payoff changes with the sizes of the two teams. If a member's payoff increases with his own team size and decreases with the size of the opponent team, then there exists a unique non-trivial equilibrium in which in the first stage the teams are stochastically formed. While this is true for a linear aggregation function, characterizing the class of functions, which induce payoffs increasing in own team size and decreasing in the opponent team size, remains an interesting open question.

Our analysis is closely related to the literature on contest theory. Most papers in this literature can be classifed according to how their models fit the following binary categorizations:

1. Who are competing: individuals or teams?
2. How is the winner chosen: stochastically or deterministically?
3. Information structure: complete or incomplete?

The literature can thus be organized into a $2 \times 2 \times 2$ design:

INDIVIDUALS	Complete	Incomplete
Stochastic		
Deterministic		

TEAMS	Complete	Incomplete
Stochastic		
Deterministic		

By now there is a vast literature that fills the cells in the "Individuals" table. Some of the prominent works in the complete information column include Hillman and Riley (1989), Baye, Kovenock, and de Vries (1996) and Siegel (2009) for the "deterministic" case and Siegel (2009) and Cornes and Hartley (2005) for the "stochastic" case . The incomplete information column includes Amann and Leininger (1996), Lizzeri and Persico (2000), Kirkegaard (2013) and Siegel (2014) for the "deterministic" case, and Ryvkin (2010), Ewerhart and Quartieri (2013) and Ewerhart (2014) for the "stochastic" case.

There is also an extensive literature on team contests with complete information. This literature includes Skaperdas (1998), Nitzan (1991), Esteban and Ray (2001, 2008), Nitzan and Ueda (2009, 2011), Münster (2007, 2009), Konrad and Leininger (2007), and Konrad and Kovenock (2009), among many others. In particular, our modeling approach of assuming that the value of winning is a public good among team members follows that of Baik, Kim, and Na (2001), Topolyan (2013) Chowdhury and Topolyan (2015) and Chowdhury, Lee, and Topolyan (2016). These studies assume that the group bid is either the minimum or the maximum of the individual bids, whereas we assume that a possible non-linear function aggregates the individual bids into a total group bid. The group size paradox fails in many of the teams-stochastic-complete models (e.g., Esteban and Ray (2001) and Nitzan and Ueda (2011)), but is satisfied in the teams-deterministic-complete model of Barbieri, Malueg, and Topolyan (2013).[1]

Our work falls into the incomplete information column, which has only been filled recently by Fu, Lu, and Pan (2015) and Barbieri and Malueg (2015). The first paper analyzes a general model that accomodates each of the cells in the "Teams" table. However, their work differs from ours in that they study a multi-battle contest: Players from two equal-size teams form pairwise matches to compete in distinct two-player all-pay auctions, and a team wins if and only if its players win a majority of the auctions. In contrast, we analyze a contest in which the members of both teams participate simultaneously in one big all-pay auction.

The second paper by Barbieri and Malueg (2015) is more closely related to our work since it also analyzes a static incomplete information (static) all-pay auction between teams that may differ in size. In contrast to us, they assume that the value of winning is an independent private value of each team member, and that the team's bid is equal to the maximal bid among its members. Under this specification they show that in the case of two teams with different cdfs, a team's probability of winning increases (decreases) with size if its cdf is inelastic (elastic). We assume that all members of a team have the same commonly known value of winning, but this value is unknown to the opponent team. As stated above, we link the size advantage/disadvantage to the curvature of the bid-aggregation function. In addition, we also analyze the implication of group size on individual welfare, and show that this also depends on the curvature of the aggregation function.

2 The Model

Two teams, B (for "big") and S (for "small"), compete for a prize. Team B has n_B players and team S has n_S players, where $n_B \geq n_S$. We denote by X a generic team and by Y the opponent team. Competition takes the following form. All players simultaneously choose some action from $\mathbb{R}^+$. Player i's chosen action e_i is interpreted as the amount of effort that player i exerts. Team X's overall performance, measured by its *score*, is given by the *aggregation function* $H\Big((e_i)_{i \in X}\Big) = \sum_{i \in X} h(e_i)$ where h is some real valued function. Assume that h is strictly increasing, twice differentiable, and normalized so that $h(0) = 0$. Clearly H is convex, concave or linear if and only if h is respectively convex, concave or linear.

The higher scoring team wins the prize. A tie is broken by a fair coin. Every

[1] The papers demonstrating the failure of the group-size paradox do so by assuming diminishing marginal team performance to the cost born by each individual. However, since these papers analyze a very different framework than ours, our result does not follow from theirs.

member of team X receives a payoff of $v_X \in [0,1]$ from winning the prize. v_X is known to members of team X before the contest starts, but is unknown to members of the other team. It is common knowledge that v_B and v_S are both drawn from the same distribution F, where F admits a strictly positive density function f. Regardless of which team wins the prize, each player pays a cost equal to the amount of effort he exerted. Thus, the net payoff to player i in team X who exerted effort e_i is $\mathbf{1}_X v_X - e_i$, where $\mathbf{1}_X$ is equal to 1 if team X won and is 0 otherwise.

3 The Analysis

In the paper we focus on pure strategy Bayesian Nash equilibria (BNE).[2] A BNE can be characterized by a vector of *effort functions* (e_i) such that $e_i(v)$ is the amount of effort player i exerts if the value of the prize is v. Given any BNE there are associated equilibrium *score functions* P_B and P_S such that $P_X(v) := \sum_{i \in X} h(e_i(v))$ is the score of team X if the value of the prize is v. A BNE is *monotone* if every player's effort is weakly increasing in his valuation of the prize. For the rest of the paper we focus on monotone BNE. It is straightforward that in a monotone BNE, P_B and P_S are weakly increasing as well.

Let G_X denote the ex ante equilibrium distribution of P_X. The following lemma extends Lemmas 1-3 and 5 of Amann and Leininger (1996) to the present setting.

Lemma 1. *For any monotone BNE:*

1. $P_B(1) = P_S(1) = P(1)$.
2. G_B *and* G_S *have a common support.* $[0, P(1)]$, *and both are continuous over this support.*
3. $\min\{G_B(0), G_S(0)\} = 0$.

Given a monotone BNE, for any team X, player $i \in X$ and value $v \in [0,1]$, $e_i(v)$ is the solution to the following maximization problem:

$$\max_{e \geq 0} G_Y\Big(\sum_{j \in X, j \neq i} h(e_j(v)) + h(e)\Big)v - e.$$

It follows from Lemma 1 that the first order condition

$$G'_Y\Big(P_X(v)\Big)h'(e_i(v))v = 1 \tag{1}$$

[2] One of the reasons we study the incomplete information model, instead of the complete information counterpart, is that pure strategy equilibria typically do not exist in the latter. On the other hand, since we impose no additional assumption on the valuation distribution F, all results in the paper hold for a model with "almost complete" information in which F is arbitrarily close to a degenerate distribution.

holds if G_Y is differentiable at $P_X(v)$ and $e_i(v) > 0$.

We say that a BNE satisfies the *active-participation* property if there does not exist a player who always exerts zero effort. We say that a BNE is *in-team symmetric* if the members of each team use the same strategy.

The following proposition describes the set of all monotone BNE for each of the three cases: (1) h is strictly concave, (2) h is strictly convex, (3) h is linear. Moreover it establishes the uniqueness and in-team symmetry of active-participation BNE in the first two cases.

Proposition 1.

1. *If h is strictly concave/convex, then there is a unique monotone BNE, and this BNE satisfies active-participation and is in-team symmetric.*
2. *If h is linear then there is a continuum of monotone BNE, of which only one is in-team symmetric and monotone. Moreover, every monotone BNE of every contest (characterized by the team size parameters (n_B, n_S)) has the same equilibrium team score functions (P_B, P_S).*

Proof. Suppose h is strictly concave or convex. Pick any two players i and j in team X. If $e_i(v) > 0$ and $e_j(v) > 0$ then equation (1) implies $h'(e_i(v)) = h'(e_j(v))$, which in turn implies $e_i(v) = e_j(v)$ because h' is strictly monotone. Thus if h is strictly concave or convex then in any monotone BNE, given any v, two players in the same team exert the same effort if neither is shirking.

Suppose h is strictly concave. Consider some team X and any v such that $e_i(v) > 0$ for some $i \in X$. Then the first order condition (1) holds for i. If there is some $j \in X$ such that $e_j(v) = 0$ then since $h'(0) > h'(e_i(v))$ we have

$$G'_Y\Big(P_X(v)\Big)h'(0) > 1,$$

implying that player j can profit by increasing his effort, a contradiction. Thus, there does not exist v such that in the same team some players work and some players shirk. Hence if h is strictly concave then any monotone BNE is in-team symmetric.

Recall that in a monotone BNE, P_X is weakly increasing. Lemma 1 implies that the distribution G_X has nop atoms, which combined with our assumption that the density f is strictly positive implies that P_X is strictly increasing on $(0, P_X(1))$. Clearly for any $t > 0$, $G_Y(t) = \Pr(P_Y(v) \leq t) = \Pr(v \leq P_Y^{-1}(t)) = F(P_Y^{-1}(t))$. Thus we have $G'_Y(P_X(v)) = f\Big(P_Y^{-1}(P_X(v))\Big)(P_Y^{-1})'(P_X(v))$. In-team symmetry implies equation (1) can be rewritten as

$$f\Big(P_Y^{-1}(P_X(v))\Big)(P_Y^{-1})'(P_X(v))h'\Big(h^{-1}(P_X(v)/n_X)\Big)v = 1. \tag{2}$$

For any v such that $P_X(v) > 0$, equation (2) can be simplied by a change of variable $t = P_X(v)$ and reduce to

$$f\Big(P_Y^{-1}(t)\Big)(P_Y^{-1})'(t)P_X^{-1}(t)h'\Big(h^{-1}(t/n_X)\Big) = 1.$$

Given Lemma 1 it is straightforward to verify that (P_B, P_S) determines a monotone BNE if and only if $P_B = \max(\beta, 0)$ and $P_S = \max(\sigma, 0)$ where (β, σ) solves the following boundary value problem:

$$f\Big(\sigma^{-1}(t)\Big)(\sigma^{-1})'(t)\beta^{-1}(t)h'\Big(h^{-1}(t/n_B)\Big) = 1 \tag{3}$$

$$f\Big(\beta^{-1}(t)\Big)(\beta^{-1})'(t)\sigma^{-1}(t)h'\Big(h^{-1}(t/n_S)\Big) = 1 \tag{4}$$

with boundary conditions:

$$\beta(1) = \sigma(1) \tag{5}$$

$$\max\{\beta(0), \sigma(0)\} = 0. \tag{6}$$

This boundary value problem is exactly the same boundary value problem that characterizes the monotone BNE of an all-pay contest between two players B and S whose valuations are independently distributed according to F, such that the score of player X is the same as his chosen amount of effort, and the cost of exerting effort e is equal to $c_X(e) := \int_0^e \frac{1}{h'\left(h^{-1}(t/n_X)\right)}$. By Proposition 1 of Kirkegaard (2013) the auxiliary two-player game has a unique monotone BNE. [3] Part 1 immediately follows.

Suppose h is strictly convex. That an in-team symmetric monotone BNE is unique is established exactly as above. Clearly that BNE satisfies active-participation. Now we show that an active-participation monotone BNE must be in-team symmetric. Pick any active-participation monotone BNE. Suppose in team X there are players i and j whose effort functions are different. Thus there exists some v such that $e_i(v) \neq e_j(v)$. In the first paragraph of this proof we showed that this implies that $e_i(v)$ and $e_j(v)$ cannot be both positive. Without loss of generality assume $e_i(v) > e_j(v)$. It follows that $e_j(v) = 0$. Since e_j is weakly increasing and is not constant, there is some $\overline{v} \geq v$ such that $e_j(\overline{v}-\epsilon) = 0$ and $e_j(\overline{v} + \epsilon) > 0$ for any $\epsilon > 0$ if $\overline{v} < 1$, or $e_j(\overline{v} - \epsilon) = 0$ and $e_j(\overline{v}) = 1$ for any $\epsilon > 0$ if $\overline{v} = 1$. Suppose $\overline{v} < 1$. We have $e_j(\overline{v} + \epsilon) = e_i(\overline{v} + \epsilon) \geq e_i(v) > 0$. It

[3] Kirkegaard (2013) imposes additional constraints on c_X, but those constraints are irrelevant for the existence and uniqueness of the solution to the boundary value problem.

follows that

$$\begin{aligned}\lim_{\epsilon\to 0}\Big(P_X(\overline{v}+\epsilon)-P_X(\overline{v}-\epsilon)\Big) &\geq \lim_{\epsilon\to 0}\Big(h(e_j(\overline{v}+\epsilon))-h(e_j(\overline{v}-\epsilon))\Big)\\ &\geq h(e_i(v))-h(0)\\ &= h(e_i(v))\\ &> 0.\end{aligned}$$

However this is a contradiction because G_X is continuous at $\overline{v}$ by Lemma 1. Similarly $\overline{v}=1$ also leads to a contradiction. Thus $e_i(v)=e_j(v)$ for any v. This establishes Part 2.

Now show Part 3. If h is linear then h' is some constant $\gamma>0$. It is easy to verify that (P_B,P_S) are monotone BNE team score functions if and only if $P_B=\max(\beta,0)$ and $P_S=\max(\sigma,0)$ where (β,σ) solves boundary value problem given by (3)-(6), where $h'(h^{-1}(t/n_X))=\gamma$. Since the BVP has a unique solution, any two monotone BNE have the same team score functions. Moreover since the BVP does not depend on (n_B,n_S), neither does its solution. ∎

In the rest of the paper we focus on active-participation BNE because we are interested in the implication of team size on performance. If a player shirks all the time, then he has no impact on the contest and is essentially absent from his team. Hence the "effective" team size should not take him into account. Proposition 1 implies that restricting attention to active-participation BNE is without loss of generality if h is concave because in this case it is the unique monotone BNE, and is without loss of "much" generality if h is linear because all monotone BNE are equivalent at the team level. However, it is worth noting that if h is convex, then there are additional monotone BNE. It is easy to verify each of those non-active-participation BNE looks exactly like the unique active-participation BNE of the smaller contest with all the shirking members removed.

3.1 Team Size and Performance

Since a team in our model is characterized by its size, a natural question that arises is whether a bigger team is more likely to win. Recall that if members of both teams exert the same amount of effort, then the bigger team has a higher score and wins. Therefore, the bigger team has a size advantage. However, because the prize is a pure public good, free-riding may be more serious in the bigger team and hence may undermine its performance. It is therefore not apriori clear which of the two forces is stronger, the size advantage or the free-riding problem.

Our next result establishes that the bigger team is more (less) likely to win if there are diminishing (increasing) returns to effort. Formally, the effect of team size on the probability of winning is determined by the curvature of h. This means that the bigger team has an advantage in situations where the biggest contribution to performance occurs early on. On the other hand, in tasks where greater expertise (which increases the rate of return) requires higher effort, the smaller team will have an advantage.

Proposition 2. *In an active-participation monotone BNE:*

1. *If h is strictly concave then $P_B(v) \geq P_S(v)$ for any v and team B is more likely to win.*
2. *If h is strictly convex then $P_B(v) \leq P_S(v)$ for any v and team S is more likely to win.*
3. *If h is linear then $P_B(v) = P_S(v)$ for any v and both teams win with the same probability.*

Proof. Fix an active-participation monotone BNE. Suppose h is strictly concave and $P_B(v) = P_S(v) = t > 0$ for some v. Thus by equations (3) and (4) we have

$$(P_S^{-1})'(t)h'\Big(h^{-1}(t/n_B)\Big) = \frac{1}{f(v)v}$$
$$(P_B^{-1})'(t)h'\Big(h^{-1}(t/n_S)\Big) = \frac{1}{f(v)v}.$$

Since h is strictly increasing, $n_B \geq n_S$ implies $h^{-1}(t/n_B) < h^{-1}(t/n_S)$, which in turn implies $h'(h^{-1}(t/n_B)) > h'(h^{-1}(t/n_S))$. Thus $(P_S^{-1})'(t) < (P_B^{-1})'(t)$, which implies that $P_S'(v) > P_B'(v)$. Since $P_B(1) = P_S(1)$ by Lemma 1(1), it follows that $v = 1$. Given Lemma 1(3), that $P_S'(1) > P_B'(1)$ then implies $P_B(v) > P_S(v)$ for any $v \in (0, 1)$. Part 1 of the present proposition immediately follows. Part 2 is established with the symmetric argument.

To show Part 3, notice that if h is linear then the boundary value problem given by equations (3)-(6) are symmetric in (β, σ). Thus the uniqueness of the solution implies that $\beta = \sigma$, in turn implying Part 3. ∎

Proposition 2 implies that the existence of the "group-size paradox" depends on whether a player's mariginal return to effort is diminishing or increasing. To give some intuition for this result consider the case of a strictly concave h.

1. If a team with n members incurs a *total cost* of C, then the team score will be $nh(C/n)$. Taking the team as a whole, the marginal productivity is thus $\frac{d}{dC}nh(C/n) = h'(C/n)$. Since h is strictly concave, it follows that for a given C, the marginal productivity of a team increases with team size. Thus given the same total cost C we have $n_B h(C/n_B) > n_S h(C/n_S)$, implying that the score of the bigger team is higher.
2. For an n-member team to achieve a total score of T, each member's effort must be $h^{-1}(T/n)$, and therefore each member's individual mariginal productivity is $(h^{-1})'(T/n))$. This individual marginal productivity is also

increasing in n. In other words, additional members make existent members more "productive": To achieve the same team score everyone on the bigger team can now decrease his effort, which simultaneously increases his marginal productivity (this effect is somewhat similar to that of strategic complementarity).

3.2 Team Size and Individual Welfare

Would a player prefer to be in the bigger team or in the smaller team? We cannot answer this question by merely comparing the winning probabilities because a higher winning probability may require a higher level of individual effort, which may offset the gain from a higher winning probability. Suppose h is either concave, convex or linear, and that the players coordinate on the unique in-team symmetric monotone BNE.[4] It is clear that the ex ante equilibrium expected payoff for a player in team X depends only on the number of players in each team. Let $u(n_X, n_Y)$ denote this expected payoff for a player in team X. The following proposition shows that if h is weakly concave, that is, if marginal returns to effort is weakly diminishing, then in equilibrium members of the bigger team are better off than members of the smaller team.

Proposition 3. *If h is weakly concave, then $u(n_B, n_S) > u(n_S, n_B)$ if $n_B > n_S$.*

Proof. Let e_X denote the effort function of a player in team X in the symmetric monotone BNE. Define $P_X^{-1}(0) = 0$. Thus

$$u(n_X, n_Y) = \int_0^1 F\Big(P_Y^{-1}(P_X(v))\Big) v dF(v) - \int_0^1 e_X(v) dF(v).$$

First suppose h is strictly concave. Thus by Proposition 2(1), $P_B(v) \geq P_S(v)$ with the inequality being strict for a set of values with positive measure under F. Suppose a player in team B unilaterally deviates to using the effort function

$$\hat{e}(v) = \max\Big\{0, h^{-1}\Big(P_S(v) - (n_B - 1)h(e_B(v))\Big)\Big\},$$

that is, the player shirks as much as he can to ensure that for any v the resulting team score $\hat{P}_B(v)$ is as high as $P_S(v)$. That $P_B(v) \geq P_S(v)$ implies $\hat{e}(v) \leq e_B(v)$. It follows from $P_B(v) \geq \hat{P}_B(v) \geq P_S(v)$ that

$$P_S^{-1}(\hat{P}_B(v)) \geq \hat{P}_B^{-1}(P_S(v)) \geq P_B^{-1}(P_S(v))$$

with at least one of the two above inequalities being strict for any v such that $P_B(v) \geq P_S(v)$.

[4] We get in-team symmetry for free by Proposition 1 if h is strictly convex or concave.

Now we show that $\hat{e}(v) \leq e_S(v)$. This is clearly true for any v such that $e_B(v) \leq e_S(v)$. For any v such that $e_B(v) \geq e_S(v)$ we have $(n_B - 1)h(e_B(v)) \geq n_S h(e_B(v)) \geq n_S h(e_S(v)) = P_S(v)$, implying $h^{-1}\Big(P_S(v) - (n_B - 1)h(e_B(v))\Big) < 0$, which in turn implies $\hat{e}(v) = 0 \leq e_S(v)$.

Thus we have

$$\begin{aligned}
u(n_B, n_S) &\geq \int_0^1 F\Big(P_S^{-1}(\hat{P}_B(v))\Big) v dF(v) - \int_0^1 \hat{e}(v) dF(v) \\
&\geq \int_0^1 F\Big(\hat{P}_B^{-1}(P_S(v))\Big) v dF(v) - \int_0^1 \hat{e}(v) dF(v) \\
&> \int_0^1 F\Big(P_B^{-1}(P_S(v))\Big) v dF(v) - \int_0^1 \hat{e}(v) dF(v) \\
&\geq \int_0^1 F\Big(P_B^{-1}(P_S(v))\Big) v dF(v) - \int_0^1 e_S(v) dF(v) \\
&= u(n_S, n_B).
\end{aligned}$$

If h is linear then $P_B = P_S$ by Proposition 2(3). Thus $v = P_S^{-1}(P_B(v)) = P_B^{-1}(P_S(v))$ and $e_B(v) \leq e_S(v)$ with the inequality being strict if $e_B(v) > 0$. Thus

$$\begin{aligned}
u(n_B, n_S) &= \int_0^1 F(v) v dF(v) - \int_0^1 e_B(v) dF(v) \\
&> \int_0^1 F(v) v dF(v) - \int_0^1 e_S(v) dF(v) \\
&= u(n_S, n_B).
\end{aligned}$$

∎

If h is convex, then the argument used in the proof of Proposition 3 does not work, and the welfare comparison in general is unclear. However, we can establish that for *some* convex h, it is still the case that each member of the bigger team receives a higher expected payoff than each member of the smaller team, even though the smaller team is more likely to win.

Proposition 4. *For any $n_B > n_S$ there exists some convex h such that $u(n_B, n_S) > u(n_S, n_B)$.*

Proof. Pick any n_B and n_S where $n_B > n_S$. Define $h^\alpha(x) = x^\alpha$. Let $v(\alpha|n_X, n_Y)$ be equal to $u(n_X, n_Y)$ for the game with $h = h^\alpha$. Clearly $v(\alpha|n_X, n_Y)$ is continuous in α. Since $v(1|n_B, n_S) > v(1|n_S, n_B)$ by Proposition 3, there exists some $\alpha > 1$ such that $v(\alpha|n_B, n_S) > v(\alpha|n_S, n_B)$. Clearly h^α is convex if $\alpha > 1$. ∎

3.3 Endogenous Team Formation

So far we assumed that the competing teams are exogenously given. In this section we investigate how teams come into being. To do this, we consider the following two-stage game.

1. In the first stage, an even number of $N > 2$ players simultaneously choose a letter R or L, interpreted as choosing to join team R or L.
2. In the second stage, players who chose the same letter form a team that competes with the players who chose the other letter, where the competition takes the contest format described in Section 2.

Assume that the value of the prize is realized only after the teams are formed. Also, if all players choose the same team, then there is no contest and the prize is awarded to that single team.

Obviously, there is always a trivial equilibrium in which everyone chooses the same team. However, this equilibrium is uninteresting, and also demands a lot of coordination from the players. We therefore explore other equilibria. To do this, assume that h is concave, convex or linear, and that in the second stage the teams coordinate on the unique in-team symmetric monotone BNE. It turns out that team formation depends on how a player's ex ante equilibrium expected payoff u changes with the teams' sizes.

Proposition 5. *If $u(n_X, n_Y) < u(n_X + 1, n_Y - 1)$ for any n_X and n_Y, then there is a unique non-trivial equilibrium in which each player chooses either team with equal probability.*

Proof. Pick any non-trivial equilibrium. We first show that every player is indifferent between choosing either letter. Suppose there is some player i who strictly prefers to choose R. Pick another player j and denote the probability that he chooses R as p. Let $r(n)$ denote the probability that n players other than i and j choose R, and $l(n)$ the analogous probability for L. Player i's expected payoff from choosing R is

$$\begin{aligned} v^R(p) &= \sum_{n=0:N-2} r(n)\Big[pu\Big(n+2, N-n-2\Big) + (1-p)u\Big(n+1, N-n-1\Big)\Big] \\ &= p \sum_{n=0:N-2} r(n)\Big[u\Big(n+2, N-n-2\Big) - u\Big(n+1, N-n-1\Big)\Big] \\ &\quad + \sum_{n=0:N-2} r(n)u\Big(n+1, N-n-1\Big) \end{aligned}$$

Likewise his expected payoff from choosing L is

$$\begin{aligned} v^L(p) &= \sum_{n=0:N-2} l(n)\Big[(1-p)u\Big(n+2, N-n-2\Big) + pu\Big(n+1, N-n-1\Big)\Big] \\ &= (1-p) \sum_{n=0:N-2} l(n)\Big[u\Big(n+2, N-n-2\Big) - u\Big(n+1, N-n-1\Big)\Big] \\ &\quad + \sum_{n=0:N-2} l(n)u\Big(n+1, N-n-1\Big) \end{aligned}$$

By assumption $u\Big(n+2, N-n-2\Big) - u(n+1, N-n-1) > 0$ for each $n = 0 : N-2$. Thus $v^R(p)$ is increasing in p and $v^L(p)$ is decreasing in p.

It is easy to verify that player j's expected payoff from choosing R is $v^R(1)$ and that from choosing L is $v^L(1)$, because player i chooses R with certainty. That player i strictly prefers R to L implies $v^R(p) > v^L(p)$, which in turn implies that $v^R(1) > v^L(1)$. Therefore player j also chooses R with certainty. It follows that every player chooses R with certainty, a contradiction because we have arrived at a trivial equilibrium.

Suppose player j chooses R with probability p and player i with probability q. Inheriting the notation from above, we have $v^R(p) = v^L(p)$ and $v^R(q) = v^L(q)$ because by the previous paragraph both i and j are indifferent between choosing either letter. Thus $p = q$ since v^R is increasing and v^L is decreasing. It follows that in the non-trivial equilibrium, for any player the following indifference condition holds:

$$\begin{aligned} &\sum_{n=0:N-1} \binom{N-1}{n} p^n (1-p)^{N-1-n} u(n+1, N-1-n) \\ = &\sum_{n=0:N-1} \binom{N-1}{n} (1-p)^n p^{N-1-n} u(n+1, N-1-n). \end{aligned}$$

That u is increasing in its first argument and decreasing in its second argument implies the left hand side is increasing in p and the right hand side is decreasing in p. Thus the only solution to this equation is $p = 0.5$. ∎

Propositions 1 and 2 imply that if h is linear then regardless of the number of players in each team, the unique symmetric monotone BNE has the same team score functions (P_B, P_S) and moreover $P_B = P_S =: P$. Thus

$$u(n_X, n_Y) = \int_0^1 \Big(F(v) - h^{-1}(P(v)/n_X)\Big) dF(v).$$

$u(n_X, n_Y)$ is strictly increasing in n_X because h is increasing, and is invariant with respect to n_Y. Thus we have the following corollary of Proposition 5:

Corollary 1. *If h is linear, then there is a unique non-trivial equilibrium in which each player chooses either team with equal probability.*

We were not able to analytically establish how u changes with the teams' sizes for more general h functions. Numerical simulations suggest that u is increasing in n_X and decreasing in n_Y also when h is strictly convex and strictly concave.[5]

4 Concluding remarks

We proposed to model competition between teams as a contest between two groups of players, where each single player incurs the cost of his own effort and the team's overall effort is some aggregation of the individual efforts of its members. What makes a collection of individuals a "team" is the fact that the award from winning is a pure public good among them, and the value of this public good is common knowledge among the members. In contrast, the value of the award to the opposing team is not observed and is treated as a random variable.

This model allowed us to analyze whether the bigger team has an advantage, and whether the members of the bigger team are better off. Our results shed new light on the "group-size paradox" by showing that the strategic effect of size depends on whether the marginal effect of individual effort is diminishing or not. We interpret this to mean that size advantage may depend on the particular task at hand, which determines how the marginal contribution of effort changes with the level of effort.

Future work should try and explore how other characteristics of teams - such as the composition of heterogenous teams, or the communication protocols among their members - affect the outcome of competition. The ultimate goal is to try and incorporate into standard models of strategic interaction the idea that the players are actually groups of individuals who have to consider the actions of their peers as well as those of the competing group.

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[5] An alternative model of group formation is one where team members sequentially decide which team to join. Although this model can be solved via backwards induction, it does require to put more structure on the function u. For the linear case, sequential participation would necessarily lead to the trivial equilibrium in which everyone joins the same team.

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