Why Voting? A Welfare Analysis

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A committee decides collectively whether to accept a given proposal or to maintain the status quo. Committee members are privately informed about their valuations and monetary transfers are possible. According to which rule should the committee make its decision? We consider strategy-proof and anonymous mechanisms and solve for the decision rule that maximizes utilitarian welfare, which takes monetary transfers to an external agency explicitly into account. For regular distributions of preferences, we find that it is optimal to exclude monetary transfers and to decide by qualified majority voting. This sheds new light on the common objection that criticizes voting for its inefficiency.
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Majority voting is inefficient from a utilitarian perspective because the decision rule does not condition on preference intensities. Consider, for example, a municipality that decides whether to adopt a new law. Suppose a majority has a weak preference against the law, but there is a minority that strongly prefers the proposed law. Even if implementing the law maximizes utilitarian welfare, the law will be defeated if the decision is made by majority voting. Can this municipality benefit from a more complex decision rule that enables voters to signal their preference intensities, for example by allowing lobbying activities or even direct payments?

A specific suggestion is to implement the efficient decision rule (i. e. the one that always chooses the decision that is best from a utilitarian perspective) by using a Vickrey-Clarke-Groves (VCG) mechanism. However, if agents are privately informed this requires that agents make payments and it is well-known that these payments cannot balance the budget. Do agents prefer the efficient rule if they have to make payments in turn? We show that whenever an anonymous decision rule conditions on preference intensities, money is lost, which introduces a trade-off for the agents: they can choose a "good" decision rule, but then they will lose money. We study this trade-off and solve for the decision rule that maximizes

utilitarian welfare in a class of plausible decision rules. Majority voting turns out to be optimal in our model.

Our analysis follows standard models of collective decision making: A finite population of voters decides collectively whether to accept a given proposal or to maintain the status quo. Agents are privately informed about their valuations and have quasi-linear utilities. Monetary transfers are feasible as long as they create no budget deficit and as long as agents are willing to participate in the decision process. In contrast to much of the literature, we consider a utilitarian welfare function that takes monetary transfers to an external agency into account. We investigate which strategy-proof and anonymous mechanism maximizes expected utilitarian welfare. Strategy-proofness ensures that the rule can be robustly implemented, while anonymity seems to be a reasonable fairness requirement for a society of equals.

Our main result provides conditions under which the optimal mechanism is implementable by qualified majority voting. Under such schemes, agents simply indicate whether they are in favor or against the proposal, and the proposal is accepted if the number of agents being in favor is above a predetermined threshold. This implies that even though it is possible to use monetary transfers, it is optimal not to use them. Specifically, we show that any anonymous decision rule that relies on monetary transfers wastes money to such an extent that, for regular distributions of types, it is inferior to voting. In our model it is therefore not possible to improve upon voting without giving up reasonable properties of the mechanism.

Our finding that voting performs well from a welfare perspective stands in sharp contrast to parts of the previous literature, which suggest implementing the value-maximizing public decision. However, this does not achieve the first-best because it induces budget imbalances (see, e.g., Green and Laffont 1979). While it is traditionally assumed that money wasting has no welfare effects, we consider a social planner that cares about aggregate transfers.[1] Our approach seems reasonable for at least two reasons: First, a utilitarian planner is interested in implementing the decision rule that maximizes the agents' expected utility. Since agents care about money, the planner in turn cares about aggregate transfers. Second, groups often choose the rule by which they decide themselves, and when making this choice they take the payments they have to make into account. Hence, our approach characterizes decision rules that are likely to prevail in practice.

Our derivation that transfer-free voting schemes dominate more complex decision rules can be summarized as follows. To prevent agents from overstating their preference intensities, one has to impose incentive payments whenever a

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[1] An early exception is Tideman and Tullock (1976), who argue that the budget imbalances of VCG mechanisms are not important because they vanish as the populations grow and hence are quantitatively negligible in many practical applications. We review their argument in Section III. Note that the nature of the objective function would be irrelevant if we considered Bayesian incentive compatible mechanisms, as the expected externality mechanism (d'Aspremont and Gerard-Varet 1979, Arrow 1979) achieves the first best in our setting with a balanced budget.

decision rule conditions on preference intensities. Strategy-proof implementation severely restricts how these payments can be redistributed to the agents: the redistribution payment an agent receives must not depend on his own reported preference. For anonymous decision rules, we show that this restriction prevents any redistribution; hence, all incentive payments that are collected to induce truthful reports have to be wasted. This implies as a corollary that an anonymous mechanisms is implementable with a balanced budget if and only if it can be implemented by qualified majority voting. The result that no money can be redistributed fixes the trade-off between increasing efficiency of the public decision and reducing the waste of monetary resources. For regular distribution functions, we show that this trade-off is solved optimally by not using money at all. This implies that the optimal mechanism is implementable by qualified majority voting. We characterize the minimum number of votes that is optimally required for the adoption of the proposal.

Related Literature

The literature evaluating public decision rules by the utilitarian criterion was initiated by Rae (1969), who compares expected welfare of different voting rules and shows that simple majority voting (where a proposal is accepted if at least half of the population votes for it) is optimal if preferences are symmetric across outcomes. Recently, this approach was generalized using insights from the mechanism design literature to include more general decision rules (Azrieli and Kim 2014), to allow for correlated valuations (Schmitz and Tröger 2012) and to consider environments with more than two alternatives (Gershkov, Moldovanu and Shi 2016). While we extend this approach to allow for monetary transfers, the resulting optimal decision rules relate our study to this literature.

Our results contribute to studies that try to explain why voting rules are used instead of mechanisms that rely on transfers: Bierbrauer and Hellwig (2016) argue that voting rules are unique in being robust and coalition-proof. Ledyard and Palfrey (2002) argue that voting rules are easy to implement and show that they approximate the efficient decision rule in large populations. In contrast, our results also apply for fixed finite populations.

In independent work, Shao and Zhou (2016*b*) analyze the optimal mechanism for the provision of a costly public good. They show that equal cost sharing mechanisms are optimal (in expectation) among dominant-strategy incentive compatible and feasible mechanisms that satisfy an additional kindness axiom. Our results are not logically related (anonymous mechanisms do not necessarily satisfy their kindness axiom, the pivot mechanism being a counterexample; a "kind" mechanism need not be anonymous on the other hand). However, our results are related in spirit: both papers show that if there is a trade-off between balancing the budget or having a more efficient decision rule, it is often preferable to balance the budget. We provide in addition results for general (and potentially correlated) type distributions and on ex-post dominance. Recently, Kuzmics and Steg (2017)

study public good provision in a related setting, but impose a stronger participation constraint. This constraint (in combination with their budget constraint) rules out VCG mechanisms and non-unanimous voting rules, which are optimal in our environment. They find that the welfare maximizing mechanism can be implemented by unanimity voting with a fixed cost sharing rule.

Our modeling approach is related to a small part of the literature, which evaluates allocation rules for the allocation of a private good according to an average efficiency criterion and considers money burning to be welfare reducing (Drexl and Kleiner 2015, Shao and Zhou 2016a, Miller 2012). An alternative criterion to evaluate allocation rules is to rank them in terms of their worst-case efficiency (see, for example, Sprumont 2013, Moulin 2010, Deb and Seo 1998).

Serizawa (1999) studies more general public good economies, in which agents have general utility functions and the public good is a continuous choice. He characterizes mechanisms that are strategy-proof, budget-balancing, and satisfy additional symmetry restrictions. Instead, we do not impose budget balance and are instead interested under what conditions the optimal mechanism will be budget balanced.

Studying mechanisms that are Bayesian incentive compatible, it has long been known that improvements over voting rules are possible: d'Aspremont and Gerard-Varet (1979) provide an efficient mechanism that is Bayesian incentive compatible and ex-post budget balanced. Therefore, the tradeoff between inefficient decisions and loss of money would not emerge if we imposed interim incentive and participation constraints instead. More recently, Lalley and Weyl (2016) propose a mechanism that approximates the first-best in Bayesian-Nash equilibrium. Our results contrast these contributions by showing that improvements over voting are often not possible if a more robust equilibrium concept is imposed.

The paper is structured as follows: We present the model in Section I, derive our main results in Section II and discuss the role of the assumptions in Section III.

I. Model

A population of n agents, $N = \{1, ..., n\}$, decides collectively on a binary outcome $X \in \{0, 1\}$. We interpret this as agents deciding whether they accept a proposal (in which case $X = 1$) or reject it and maintain the status quo ($X = 0$). Given a collective decision X, the utility of agent i is given by $\theta_i \cdot X + t_i$, where θ_i is the agent's valuation for the proposal and t_i is a transfer to agent i.[2] Each agent is privately informed about his valuation, which is drawn from a type space

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[2]Our analysis applies to costless projects as well as to costly projects with a given payment plan, in which case the valuation of agent i is interpreted as her net valuation taking her contribution into account. Also note that the analysis accommodates more general utility functions: Take any quasi-linear utility function such that the utility difference between $X = 1$ and $X = 0$ is continuous and strictly increasing in θ_i. Redefining the type to equal the utility difference, we can proceed with our analysis without change.

$\Theta := \left[\underline{\theta}, \overline{\theta}\right]$. To make the problem interesting we assume that $\underline{\theta} < 0 < \overline{\theta}$. Let Θ^n denote the product type space consisting of complete type profiles with typical element $\theta = (\theta_i, \theta_{-i})$.

A mechanism in this setting determines for which preference profiles the proposal is accepted and which transfers are made to the agents. Invoking the revelation principle (Gibbard 1973), we focus on direct mechanisms. Formally, a *mechanism* is a pair (x, t) consisting of a *decision rule*

$$x \ : \ \Theta^n \ \rightarrow \ \{0, 1\}$$

and a *transfer rule*

$$t : \Theta^n \rightarrow \mathbb{R}^n$$

such that, for any realized preference profile θ, $x(\theta)$ is the decision on the public outcome and $t_i(\theta)$ is the transfer received by agent i.

Not all mechanisms can plausibly be implemented. In the following, we describe several restrictions on the mechanisms that we consider.

We require that mechanisms are *feasible* in the sense that for any realization of preferences no injection of money from an external agency is necessary, i.e., for all $\theta \in \Theta^n$,

(F) $$\sum_{i \in N} t_i(\theta) \le 0.$$

Given that preferences are observed privately by the agents, a mechanism must induce the agents to report their types truthfully. We are interested in mechanisms that are *strategy-proof*: for every i, θ_i, θ_i', and θ_{-i},

$$\theta_i x(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i}) \ge \theta_i x(\theta_i', \theta_{-i}) + t_i(\theta_i', \theta_{-i}).$$

This is a strong equilibrium concept that ensures that the mechanism is incentive compatible implemented irrespective of the exact information structures. Requiring strategy-proofness is a standard approach in social choice theory (see, e.g., Moulin 1983) and implies robust implementation in the spirit of Bergemann and Morris (2005) (see Bierbrauer and Hellwig 2016).

In many situations agents have the outside option to abstain from the decision process and leave the decision to the other agents. In this case, they cannot be forced to make payments that relate to the decision process; however, agents are nonetheless affected by the public outcome. Therefore, the choice of the outcome if agent i does not participate in the decision process becomes part of the design problem. Formally, the designer can choose a tuple of functions $\left\{(\underline{x}^i, \underline{t}^i)\right\}_{i \in N}$ with $\underline{x}^i : \Theta^{-i} \rightarrow \{0, 1\}$ and $\underline{t}^i : \Theta^{-i} \rightarrow \mathbb{R}^n$ such that $\underline{t}_i^i \equiv 0$, which specify the outcome

in case agent i abstains.[3] We say that a mechanism (x, t) satisfies *universal participation* if there exists a tuple $\{(\underline{x}^i, \underline{t}^i)\}_{i \in N}$ such that all agents participating in the mechanism (x, t) is an ex-post equilibrium: for all $i \in N$ and $\theta \in \Theta^n$,

(UP) $$\theta_i x(\theta) + t_i(\theta) \geq \theta_i \underline{x}^i(\theta_{-i}),$$

and we impose that mechanisms satisfy universal participation.[4] Note that by giving the designer freedom in choosing $\{(\underline{x}^i, \underline{t}^i)\}$ we allow the designer to choose a public outcome trying to punish a given agent for non-participation, which makes the participation constraint potentially weaker. We will show below (see the proof of Lemma 2) that the designer is not going to abuse this freedom in choosing implausible punishment schemes; to induce participation it will be sufficient to set $\underline{x}_i(\theta_{-i}) = x(0, \theta_{-i})$. This participation constraint turns out to be weaker than the requirement that every agent derive utility of at least zero (often called individual rationality) and better suited for public good environments (see Green and Laffont 1979). For instance, majority voting and the pivotal mechanism satisfy universal participation, but in general are not individually rational.[5]

We also require mechanisms to be anonymous:

Definition 1. *We call a mechanism (x, t) anonymous if the decision rule is independent of the agents' identities, i. e., if for each permutation $\pi : N \to N$ and corresponding function $\hat{\pi}(\theta) = (\theta_{\pi(1)}, \ldots, \theta_{\pi(n)})$, it holds that $x(\theta) = x(\hat{\pi}(\theta))$ for all θ.*

This is a weak notion of anonymity, requiring only that the names of the agents do not affect the public decision. One reason to impose anonymity is that many fairness concepts build on this assumption (e. g., equal treatment of equals). This requirement has a long tradition in social choice theory; see for example Moulin (1983).[6]

Throughout the paper we focus on feasible and strategy-proof mechanisms that are anonymous and satisfy universal participation.

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[3]Note that we do not impose any restrictions on these functions. We will show below that the designer can choose these functions without loss to be strategy-proof and anonymous.

[4]We will show in footnote 7 how any mechanism satisfying universal participation can be extended so that participation becomes a dominant strategy.

[5]The unique voting rule that is individually rational is unanimity voting where the proposal is accepted if and only if all agents vote for the proposal. Our analysis implies that it is optimal to decide by unanimity voting if we impose individual rationality instead.

[6]Note that this assumption would be without loss of generality if we allowed for stochastic decision rules. Given any mechanism (x, t), apply this function after randomly permuting the agents. This defines a new mechanism $(\tilde{x}, \tilde{t})$ that is anonymous and achieves the same utilitarian welfare. While this new rule treats all agents equally ex-ante, it is possible that agents with the same valuations are treated very differently ex-post. Experimental evidence suggests that agents value not only ex-ante fairness, but also ex-post fairness (Brock, Lange and Ozbay 2013, Cappelen et al. 2013), which makes mechanisms that are deterministic and anonymous attractive.

II.   Results

In this section, we characterize strategy-proof mechanisms and derive an important auxiliary result about redistribution payments (Section II.A). Using this result, we show that simple majority voting is an ex-post dominant mechanism and that qualified majority voting maximizes ex-ante expected welfare (Section II.A).

A.   Characterization of mechanisms

The following lemma is a standard characterization of strategy-proof mechanisms and we omit its proof.

Lemma 1. *A mechanism is strategy-proof if and only if, for each agent i,*

1) $x(\theta_i, \theta_{-i})$ is non-decreasing in θ_i for all θ_{-i} and

2) there exists a function $h_i(\theta_{-i})$, such that for all θ,

$$(1) \qquad t_i(\theta_i, \theta_{-i}) = -\theta_i x(\theta_i, \theta_{-i}) + \int_0^{\theta_i} x(\beta, \theta_{-i}) d\beta + h_i(\theta_{-i}).$$

We call the first two terms on the right-hand side the "incentive payment" and the last term "redistribution payment".

Equation (1) suggests the following definition:

Definition 2. *Agent i is* pivotal *at profile θ if there is an $\bar{\varepsilon} > 0$ such that $x(\theta) \neq x(\varepsilon, \theta_{-i})$ for all $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$.*

If agent i is not pivotal at a given profile (θ_i, θ_{-i}) then her payment equals $h_i(\theta_{-i})$: for any $\bar{\varepsilon} > 0$ there exists $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$ such that $x(\theta) = x(\varepsilon, \theta_{-i})$; since $x(\cdot, \theta_{-i})$ is non-decreasing this implies $\theta_i x(\theta_i, \theta_{-i}) = \int_0^{\theta_i} x(\beta, \theta_{-i}) d\beta$. If an agent is pivotal, her transfer is reduced by the incentive payment.

The following lemma implies that redistribution payments are equal to 0 for the mechanisms under consideration.

Lemma 2. *Consider a strategy-proof and anonymous mechanism (x, t). (x, t) is feasible and satisfies universal participation if and only if $h_i(\theta_{-i}) = 0$ for all i and θ_{-i}.*

To obtain that $h_i(\theta_{-i})$ is necessary, we use in a first step anonymity to argue that for any profile of reports θ_{-i}, there is a report by agent i such that no agent is pivotal. This implies that all redistribution payments are zero: Suppose for a contradiction that there is a report profile θ_{-i} such that agent i receives a strictly positive redistribution. From the previous step we know that there is a profile (θ_i, θ_{-i}) such that no agent makes an incentive payment, and since the

redistribution payment of agent i does not depend on θ_i (Lemma 1), it is strictly positive at this profile. However, due to universal participation each redistribution payment has to be weakly positive and feasibility requires that their sum is weakly negative, which yields the desired contradiction.

PROOF:

Suppose $h_i(\theta_{-i}) = 0$ for all i and θ_{-i}. Then $t_i(\theta) \leq 0$ for all i and θ, and hence (x, t) is feasible. Defining $\underline{x}^i(\theta_{-i}) := x(0, \theta_{-i})$ and $\underline{t}^i(\theta_{-i}) := t(0, \theta_{-i})$ for each i and θ_{-i} and using incentive compatibility, we get

$$\theta_i x(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i}) \geq \theta_i x(0, \theta_{-i}) + t_i(0, \theta_{-i}) = \theta^i \underline{x}^i(\theta_{-i}).$$

Hence, (x, t) satisfies universal participation.

The proof of the converse consists of two steps. Suppose (x, t) is feasible and satisfies universal participation.

Step 1: For all i and θ_{-i}, there exists θ_i such that no agent is pivotal at (θ_i, θ_{-i}).

Note first that all agents that are pivotal at profile θ submit reports of the same sign: If $x(\theta) = 1$ then monotonicity implies that $x(0, \theta_{-i}) = 1$ for all agents i with $\theta_i < 0$ and hence only agents with positive reports can be pivotal (and similarly for $x(\theta) = 0$).

Fix an arbitrary agent i and a report profile $\theta_{-i} \in \Theta^{n-1}$. Suppose $x(0, \theta_{-i}) = 1$ and hence that all pivotal agents submit positive reports (if no agent is pivotal at this profile, we are done; if $x(0, \theta_{-i}) = 0$ analogous arguments hold). We show that no agent is pivotal at profile $\theta := (\theta_{j^*}, \theta_{-i})$, where $j^* \in \arg\max_j \theta_j$. Monotonicity implies that $x(\theta) = x(0, \theta_{-i}) = 1$ and hence agent i is not pivotal. Anonymity implies that agent j^* is not pivotal.

It remains to show that if j is not pivotal at θ and $\theta_{j'} \leq \theta_j$, then j' is not pivotal at θ. Assume to the contrary that j' is pivotal at θ; hence, $x(\theta) = 1$ and $x(\varepsilon, \theta_{-j'}) = 0$ for some $\varepsilon > 0$. If $\hat{\pi}_{j,j'} : \Theta^n \to \Theta^n$ is the function permuting the j-th and j'-th component, then $\hat{\pi}_{j,j'}[(\varepsilon, \theta_{-j})] \leq (\varepsilon, \theta_{-j'})$. From monotonicity it follows that $x\left(\hat{\pi}_{j,j'}[(\varepsilon, \theta_{-j})]\right) = 0$ and anonymity implies that $x(\varepsilon, \theta_{-j}) = 0$, contradicting the assumption that j is not pivotal at θ.

Step 2: For all i and θ_{-i} we have $h_i(\theta_{-i}) = 0$.

Universal participation immediately implies that an agent with valuation 0 gets a weakly positive utility: $0 \cdot x(0, \theta_{-i}) + t_i(0, \theta_{-i}) \geq 0 \cdot \underline{x}(\theta_{-i})$. From (1) it follows that $h_i(\theta_{-i}) \geq 0$ for all i, θ_{-i}. To obtain a contradiction, suppose that there exists an agent j and a report profile $\theta_{-j} \in \Theta^{n-1}$ such that $h_j(\theta_{-j}) > 0$. By step one, we can choose θ_j such that no agent is pivotal at $\theta := (\theta_j, \theta_{-j})$, implying by (1) that $\sum_i t_i(\theta) = \sum_i h_i(\theta_{-i}) > 0$, which contradicts feasibility.

For any strategy-proof and anonymous mechanism (x, t) satisfying $h_i(\theta_{-i}) = 0$ for all i and θ_{-i}, we will from now on assume implicitly that any non-participating agent is treated as if she reported 0. Therefore, participating in the mechanism

is essentially a dominant strategy.[7]

Majority Voting

An indirect mechanism is called *qualified majority voting (with threshold k)*, if each agent has the message set {*yes, no*} and the proposal is implemented if and only if at least k agents send message *yes* and no monetary transfers are made, i. e., $t_i(\theta) = 0$ for all i and θ. It is called *simple majority voting* if $k = \frac{n}{2}$.

As an easy consequence, Lemma 1 and Lemma 2 permit a characterization of the set of mechanisms that have a balanced budget.

Corollary 1. *A feasible, strategy-proof, and anonymous mechanism satisfying universal participation has a balanced budget if and only if it is implementable by qualified majority voting.*

PROOF:

By Lemmas 1 and 2, the budget is balanced if and only if no agent is pivotal. This implies that the decision rule is constant in the interior of each orthant. Consequently, the decision rule can be implemented via qualified majority voting.

A related result has been obtained by Laffont and Maskin (1982), who in addition require weak Pareto efficiency but do not impose participation constraints.

Definition 3. *A mechanism (x, t) dominates another mechanism (x', t') if, for every θ, a majority of the agents prefers the outcome of mechanism (x, t) compared to the outcome of mechanism (x', t'). Formally, for all θ,*

$$\#\{i | \theta_i x(\theta) + t_i(\theta) \geq \theta_i x'(\theta) + t_i'(\theta)\} \geq \frac{n}{2}.$$

Groves and Ledyard (1977) showed that simple majority voting dominates the pivotal mechanism in a public good setting with logarithmic utilities and Ledyard (2006) obtained the same result for the model we study. Lemma 2 also allows us to extend this result to a much larger class of mechanisms.

Corollary 2. *Let (x, t) be any feasible and strategy-proof mechanism that is anonymous and satisfies universal participation. Then simple majority voting dominates (x, t).*

PROOF:

Lemma 2 implies that $t_i(\theta) \leq 0$ for all i and θ. Under simple majority voting, there is always a majority that gets its preferred alternative. These agents are weakly worse off under mechanism (x, t), because they make weakly positive payments and potentially get their less preferred alternative.

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[7]Formally, if a group $C \subseteq N$ abstains, the mechanism (x^C, t^C) defined by $x^C(\theta_{-C}) = x(0, \theta_{-C})$ and $t^C(\theta_{-C}) = t(0, \theta_{-C})$ is used for the participating agents. This implies in particular that the mechanisms used if not all agents participate are strategy-proof and anonymous.

Corollary 2 shows that if agents have precise information about each other when choosing among feasible, anonymous, and strategy-proof mechanisms that satisfy universal participation, then simple majority voting is always weakly preferred by a majority. For some settings, this might explain why simple majority voting is used.

Utilitarian Social Planner

The above result takes an ex-post dominance perspective and therefore does not take preference intensities into account. Even though there will always be a majority that ex-post prefers the outcome of simple majority voting, this does not imply that agents would choose majority voting from an ex-ante viewpoint (because the minority preferring a different mechanism might have stronger preferences). Therefore we take an ex-ante perspective in this section and study a utilitarian planner who chooses a mechanism to maximize expected utilitarian welfare given by

$$U\left(x,t\right) := \mathbb{E}_\theta \left[\sum_{i=1}^{N} \left[\theta_i x(\theta) + t_i(\theta) \right] \right].$$

The expectation is taken with respect to the beliefs of the planner about θ. We assume the planner believes that types are independently distributed according to a distribution function F which admits a strictly positive density f. Note that the planner's beliefs do not affect the incentives of the agents (as we focus on robust implementation), but only how she evaluates different rules. A mechanism is *optimal* if it maximizes this expression.

We concentrate first on distribution functions that satisfy the following condition:

Definition 4. *A distribution function F has* increasing hazard rates *if $\frac{f(\theta_i)}{1-F(\theta_i)}$ is non-decreasing for $\theta_i \geq 0$ and $-\frac{f(\theta_i)}{F(\theta_i)}$ is non-decreasing for $\theta_i \leq 0$.*

This assumption is well-known from the literature on optimal auctions and procurement auction design; it is satisfied by many commonly employed distribution functions, for example by the uniform, (truncated) normal, and exponential distributions.

We are now ready to state our main result.

Theorem 1. *Suppose F has increasing hazard rates and consider the class of feasible and strategy-proof mechanisms that are anonymous and satisfy universal participation. The optimal mechanism in this class is implementable by qualified majority voting with threshold $\lceil k \rceil$, where*

$$k := \frac{-n\ \mathbb{E}[\theta_i \mid \theta_i \leq 0]}{\mathbb{E}[\theta_i \mid \theta_i \geq 0] - \mathbb{E}[\theta_i \mid \theta_i \leq 0]}.$$

That is, the optimal decision rule does not rely on monetary transfers at all and can be implemented using a simple indirect mechanism where each agent indicates whether she is in favor or against the proposal. The proposal is accepted if more than $\lceil k \rceil$ voters are in favor.[8]

The following example illustrates how voting mechanisms compare to the first-best and the pivotal mechanism.

Example 1. *Let $n = 2$ and θ_i be independently and uniformly distributed on $[-3, 3]$ for $i = 1, 2$.*

If valuations were publicly observable, the first-best could be implemented which would yield welfare $U_{FB} = \frac{1}{2}\mathbb{E}[\theta_1 + \theta_2 \mid \theta_1 + \theta_2 \geq 0] = 1$. The best mechanism that decides efficiently is the pivotal mechanism, where each agent pays the externality she creates on other agents. It gives a welfare of $U_{VCG} = \frac{1}{2}$ (see the Appendix). Unanimity voting, that is, accepting the proposal if and only if both agents have a positive valuation, is an optimal voting rule (as is the voting rule that rejects the proposal if and only if both agents have a negative valuation). These rules yield welfare $U_{UV} = \frac{1}{4}\mathbb{E}[\theta_1 + \theta_2 \mid \theta_1 \geq 0, \theta_2 \geq 0] = \frac{3}{4}$. Hence, the welfare loss due to private information is twice as large under the best VCG mechanism as compared to unanimity voting.

The role of the underlying assumptions is discussed in Section III and a formal proof for Theorem 1 is provided in the Appendix. In the following, we build some intuition for this result.

Lemma 2 shows that money cannot be redistributed in strategy-proof mechanisms that are feasible, anonymous, and satisfy universal participation, and hence there is a direct trade-off between improving the decision rule and reducing the outflow of money. We show that under increasing hazard rates, this conflict is resolved optimally in favor of no money burning. To gain some intuition, fix a type profile of the other agents, θ_{-i}. Strategy-proofness implies that there is a cutoff θ_i^* such that the proposal will be accepted if the type of agent i is above θ_i^*. To solve for the optimal decision rule we need to find the optimal cutoff. Assume that $\sum_{j \neq i} \theta_j + \theta_i^* < 0$ and $\theta_i^* > 0$. Marginally increasing the cutoff leads to a rejection of the proposal which in this case increases efficiency (with a positive effect on welfare proportional to $f(\theta_i^*)$). On the other hand, strategy-proofness implies that agents with a type above the cutoff make a payment equal to the cutoff. Increasing the cutoff increases these payments (with a corresponding negative effect on welfare proportional to $1 - F(\theta_i^*)$). Monotone hazard rates imply that if the positive effect outweighs the negative effect at θ_i^* and if it is therefore beneficial to marginally increase the cutoff, then it is optimal to set the cutoff to the highest possible value. Symmetric arguments imply that it is optimal to set all cutoffs either equal to zero or to the boundary of the type space, and hence that the optimal mechanism can be implemented by a voting rule.

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[8]This indirect implementation also alleviates the commitment problem of the planner: Given the information she obtains in this mechanism, the decision rule promised to the agents is optimal.

The optimal number of votes required in favor of a proposal is given by the smallest integer number k such that the expected aggregate welfare of a proposal, given that k out of n voters have a positive valuation, is positive. Hence, the optimal threshold required for qualified majority voting depends on the conditional expected values given that the valuation is either positive or negative. Simple majority voting is optimal if valuations are distributed symmetrically around 0. If, however, opponents of a proposal are expected to have a stronger preference intensity, then it is optimal to require a qualified majority that is larger than simple majority.

General distributions and correlated types

In this section we generalize our analysis in two directions. First, we allow for more general distribution functions, not only those having increasing hazard rates. Second, we relax the assumption that the planner knows perfectly the type distribution. If types are drawn independently conditionally on some unknown state of the world, but the distribution depends on the state of the world, this potentially creates correlated types from the planner's perspective. Therefore, we first state the general optimization problem allowing for correlated types. Lemma 2 still applies and shows that all redistribution payments are equal to zero. Hence, in analogy to Lemma 3 in the Appendix, we can state the problem as

$$\max_{0 \leq x \leq 1} \int \Big[\sum_i \Psi(\theta_i|\theta_{-i}) \Big] x(\theta) dG(\theta)$$

s. t. x being point-wise non-decreasing,

where G denotes the joint distribution function (and g its density) and

$$\Psi(\theta_i|\theta_{-i}) = \begin{cases} -\frac{G(\theta_i|\theta_{-i})}{g(\theta_i|\theta_{-i})} & \text{if } \theta_i < 0 \\ \frac{1-G(\theta_i|\theta_{-i})}{g(\theta_i|\theta_{-i})} & \text{if } \theta_i \geq 0. \end{cases}$$

If types are independently distributed (that is, if $G(\theta) = \prod_i F(\theta_i)$), and if F has decreasing hazard rates, then the monotonicity constraint is not binding and consequently the efficient decision rule is optimal and can be implemented by the pivot mechanism. More generally, as long as types are independently distributed, a standard ironing procedure can be used to determine the optimal decision rule.

While standard procedures cannot be applied if types are not independently distributed, we can immediately deduce the optimal mechanism in two special cases. First, if the conditional hazard rates are decreasing ($-\frac{G(\theta_i|\theta_{-i})}{g(\theta_i|\theta_{-i})}$ and $\frac{1-G(\theta_i|\theta_{-i})}{g(\theta_i|\theta_{-i})}$ are non-decreasing for each θ_{-i}) and types are affiliated, then the pivotal mechanism

is optimal. Affiliation implies that an increase in θ_i increases $\Psi(\theta_j|\theta_{-j})$ for all j (see, for example, Segal 2003), and hence that the monotonicity constraint is not binding. Analogously, one can show that if the conditional hazard rates are increasing and types are negatively affiliated, then a voting mechanism is optimal.

These results are not fully satisfying because both, negatively affiliated types and decreasing hazard rates, are strong assumptions. If types are positively affiliated and the conditional distributions have increasing hazard rates, then the optimal mechanism usually depends on the details of the distribution function. To gain additional insights, we analyze the problem that arises if the planner has imperfect information about the distribution from which types are drawn. Specifically, we assume that there are finitely many states of the world, $\omega \in \Omega$, and that types are drawn independently conditional on the state ω from a distribution function F_ω.

Corollary 3. *Suppose that F_ω has increasing hazard rates and that $\mathbb{E}_{F_\omega}[\theta_i|\theta_i > 0] = \mathbb{E}_{F_{\omega'}}[\theta_i|\theta_i > 0]$ and $\mathbb{E}_{F_\omega}[\theta_i|\theta_i < 0] = \mathbb{E}_{F_{\omega'}}[\theta_i|\theta_i < 0]$ for all $\omega, \omega' \in \Omega$. Then qualified majority voting is optimal among all feasible and strategy-proof mechanisms that are anonymous and satisfy universal participation.*

Under the assumptions made in the corollary, uncertainty about the state of the world only affects the expected number of supporters of status quo, but not the expected type of a supporter of status quo. Consequently, in each state of the world the same qualified majority rule is optimal and therefore it is optimal ex-ante.

Proposition 1. *Suppose there are two states of the world, ω_1 and ω_2, each occurring with strictly positive probability. Suppose that $\mathbb{E}_{F_{\omega_1}}[\theta_i] > 0 > \mathbb{E}_{F_{\omega_2}}[\theta_i]$, and that the expected number of supporters of status quo is the same in both states. Then, for large enough populations, the pivot mechanism achieves a higher expected welfare than any qualified majority voting rule.*

If the state of the world influences the type distributions, but not the expected number of supporters of reform, the optimal majority requirement differs between the states of the world. Because the planner cannot choose the correct majority requirements, welfare under any voting procedure does not converge to first-best welfare. Because the welfare under the pivot mechanism converges to the first-best, it outperforms any voting rule for large populations.

PROOF:

We show first that the aggregate expected payment T^N converges to zero in state ω_1 (the arguments are analogous for state ω_2): Let z be the sum of $N-1$ random variables that are independently distributed according to F, and let $\tilde{F}$

be the distribution function of z. Then

$$0 \geq T^N = N \int_0^{\bar{\theta}} \int_{-\theta_i}^0 z d\tilde{F}(z) dF(\theta_i) - N \int_{\underline{\theta}}^0 \int_0^{-\theta_i} z d\tilde{F}(z) dF(\theta_i)$$

$$(2) \qquad \geq \int_0^{\bar{\theta}} -N\theta_i \tilde{F}(0) dF(\theta_i) - \int_{\underline{\theta}}^0 -N\theta_i \tilde{F}(\underline{\theta}) dF(\theta_i),$$

where the first line uses the definition of payments in the pivotal mechanism. Since $\mathbb{E}_{F_{\omega_1}}[\theta_i] > 0$, we get $N\tilde{F}(0) \leq N\mathbb{P}\Big(\big| z - (N-1)\mathbb{E}_{F_{\omega_1}}[\theta_i] \big| > \frac{1}{2}(N-1)\mathbb{E}_{F_{\omega_1}}[\theta_i] \Big)$. Since $\mathbb{E}_{F_{\omega_1}}[|\theta_i|^3] < \infty$, Theorem 1 in Katz (1963) implies that the right-hand side converges to zero as $N \to \infty$, and hence that the first term in (2) converges to zero. An analogous argument implies that the second term converges to zero. Therefore, welfare under the pivotal mechanism converges to the first-best welfare as the population grows.

For a voting procedure to approach the first-best, it must implement reform with probability approaching one in state ω_1 and with probability approaching zero in state ω_2. However, for an arbitrary voting rule the probability of implementing the reform is the same in both states of the world because the expected number of supporters of status quo is the same in both states. Consequently, no voting rule approximates the first-best in this environment and if the population is large enough, the pivotal mechanism outperforms any voting mechanism.

Green and Laffont (1977) showed that the aggregate payments in the pivotal mechanism converge to zero if the expected value of the types is different from zero and if the density of $\tilde{F}$ converges uniformly to zero sufficiently fast. More recently, Ledyard (2006) also showed that aggregate payments converge to zero in a model where the distribution of types depends on the population size and in the limit is supported on (a subset of) the positive real numbers.

III. Discussion

The fact that the efficient decision rule cannot be implemented with a balanced budget introduces a trade-off for a utilitarian planner: Should she choose a more efficient decision rule or one that requires smaller payments by the agents? We model this trade-off explicitly and solve for the welfare maximizing mechanism. For regular type distributions, it is optimal not to waste any monetary resources and to decide by majority voting.

The characterization of majority voting as the optimal mechanism relies on the specifics of our environment. In particular, the result that no money can be redistributed hinges on the anonymity requirement. If we relax this requirement (or allow for stochastic decision rules), majority voting is no longer optimal in general. The best budget-balanced mechanism for distributions that are symmetric around 0 is a "sampling Groves approach": pick a default agent, implement the

efficient decision for the remaining agents, and award all incentive payments to the default agent (Laffont and Maskin 1982).

Example 2 (cont.)**.** *Let $n = 4$ and θ_i be independently and uniformly distributed on $[-3, 3]$.*

By Theorem 1, the optimal deterministic and anonymous mechanism is given by an optimal voting rule (which accepts the proposal if either at least 2 or at least 3 agents are in favor). This yields a welfare of $U_{MV} = \frac{36}{32}$.

A sampling Groves scheme would, for example, implement the decision that is jointly optimal for agents 1,2, and 3. All incentive payments that are collected from these agents would then be awarded to agent 4. This yields welfare $U_{sGroves} = \frac{39}{32}$. If stochastic mechanisms are allowed, the same welfare can be achieved via an anonymous mechanism, that permutes the names of the agents at random and then applies the mechanism described above.

In contrast to the anonymity (respectively, non-randomness) requirement, the participation constraint seems not to be a driving force of our results. Imposing universal participation simplifies the analysis and allows for the clear-cut result that no redistribution is possible for anonymous decision rules. Without it, a characterization of the optimal redistribution payments is hard; our numerical results suggest nonetheless that voting is often optimal even if one does not impose participation constraints.

Considering a richer set of possible alternatives, the results would depend on the specification of agents' preferences. While in many cases similar trade-offs as in our model are present, our results do not extend in general; for example, in an environment with quadratic utilities and a continuum of alternatives, the efficient decision rule can be implemented with a balanced budget (Groves and Loeb 1975).

In an early contribution, Tideman and Tullock (1976) argued informally that implementing the pivotal mechanism might be a welfare-superior way to decide on public projects, even if the payments that accrue in the decision process are wasted. They suggest that aggregate payments vanish as the number of agents gets large and therefore the pivotal mechanism approximates the first best in large populations.[9] Our main result contrasts with this suggestion: we show that the optimal qualified majority voting rule is often welfare-superior to the pivot mechanism. An explanation for our result is that the optimal voting rule also approximates the first best in large populations, and therefore the pivotal mechanism does not necessarily provide higher welfare; indeed, for any finite population it often achieves a higher welfare. Our results thereby shed a new light on the widespread criticism that voting is inefficient: despite sometimes imposing the "wrong" decision, it can be optimal to ban monetary transfers and

[9]More recently, Ledyard (2006) concluded in a related setting that the pivotal mechanism produces higher welfare than simple majority voting for large populations if the distribution is sufficiently asymmetric.

decide by majority voting.

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APPENDIX

VERIFICATION OF EXAMPLE 1:

Welfare of the pivot mechanism can be expressed as the difference between the welfare of the first-best and the transfers needed to implement the efficient

decision:

$$U_{VCG} = U_{FB} - \frac{4}{36} \int_{-3}^{0} \int_{0}^{-\theta_1} (-\theta_2)\, d\theta_2\, d\theta_1 = \frac{1}{2}$$

Here, we used the fact that transfers are symmetric in the four regions $\{\theta \mid \theta_i \geq 0, \theta_j \leq 0, \theta_i + \theta_j \lesseqqgtr 0\}$ and zero everywhere else.

The following lemma shows how utilitarian welfare of a mechanism can be expressed as the sum of two terms. The first only depends on the decision rule, and the second consists of the redistribution payments.

Lemma 3. *Let (x, t) be an incentive compatible mechanism and define*

$$\text{(A.1)} \qquad \psi(\theta_i) = \begin{cases} \frac{-F(\theta_i)}{f(\theta_i)} & \text{if } \theta_i \leq 0, \\ \frac{1-F(\theta_i)}{f(\theta_i)} & \text{otherwise.} \end{cases}$$

Then we have

$$U(x, t) = \int_{\Theta^n} \left[\sum_{i \in N} \psi(\theta_i) \right] x(\theta) dF^n(\theta) + \sum_{i \in N} \int_{\Theta^{n-1}} h_i(\theta_{-i}) dF^{n-1}(\theta_{-i}).$$

PROOF:

Note that for all θ_{-i},

$$\int_{\underline{\theta}}^{\overline{\theta}} \left[\int_0^{\theta_i} x(\beta, \theta_{-i}) d\beta \right] f(\theta_i) d\theta_i$$

$$= \left[\int_0^{\overline{\theta}} x(\beta, \theta_{-i}) d\beta \underbrace{F(\overline{\theta})}_{=1} - \int_0^{\underline{\theta}} x(\beta, \theta_{-i}) d\beta \underbrace{F(\underline{\theta})}_{=0} \right] - \int_{\underline{\theta}}^{\overline{\theta}} x(\theta_i, \theta_{-i}) F(\theta_i) d\theta_i$$

$$= \int_0^{\overline{\theta}} \frac{1 - F(\theta_i)}{f(\theta_i)} x(\theta_i, \theta_{-i}) dF(\theta_i) + \int_{\underline{\theta}}^{0} \frac{-F(\theta_i)}{f(\theta_i)} x(\theta_i, \theta_{-i}) dF(\theta_i)$$

$$\text{(A.2)} \quad = \int_{\underline{\theta}}^{\overline{\theta}} \psi(\theta_i)\, x(\theta_i, \theta_{-i}) dF(\theta_i),$$

where the first equality follows from integrating by parts, the second from rearranging terms and the third from the definition of Ψ.

Now rewrite

$$U(x,t) = \int_{\Theta^n} \sum_{i \in N} \left[\theta_i x(\theta) + t_i(\theta) \right] dF^n(\theta)$$

$$= \sum_{i \in N} \int_{\Theta^{n-1}} \int_{\underline{\theta}}^{\overline{\theta}} \left[\int_0^{\theta_i} x(\beta, \theta_{-i}) d\beta + h_i(\theta_{-i}) \right] dF(\theta_i) dF^{n-1}(\theta_{-i})$$

$$= \int_{\Theta^n} \left[\sum_{i \in N} \psi(\theta_i) \right] x(\theta) dF^n(\theta) + \sum_{i \in N} \int_{\Theta^{n-1}} h_i(\theta_{-i}) dF^{n-1}(\theta_{-i}),$$

where the first equality follows by definition, the second from Lemma 1 and the third by plugging in equation (A.2).

For any subset $S \subseteq N$ of the agents, define the corresponding *orthant* as $\mathcal{O}_S = \{\theta \in \Theta^n \mid \theta_i \geq 0 \text{ if } i \in S, \theta_i \leq 0 \text{ if } i \notin S\}$.

Lemma 4. *Suppose that $\psi(\theta)$ is non-increasing in θ and $\int \psi(\theta) dF^n(\theta) < \infty$. Let $\mathcal{O}_S$ be the orthant corresponding to some subset of agents S. Then the problem*

$$\max_x \int_{\mathcal{O}_S} \psi(\theta) \cdot x(\theta) dF^n(\theta)$$

$$\text{s.t. } x \text{ is non-decreasing in } \theta$$

$$0 \leq x(\theta) \leq 1$$

is solved optimally either by setting $x^*(\theta) = 1$ or $x^*(\theta) = 0$.

The objective is to find a non-decreasing function that maximizes the integral over the product of this function with a non-increasing function. Extending Chebyshev's inequality to multiple dimensions yields that the objective function is maximized by choosing the non-decreasing function to be constant. Note that Chebyshev's inequality was introduced in a related setting by Shao and Zhou (2016*a*).

PROOF:

Suppose to the contrary that there exists a function $\hat{x}(\theta)$ that achieves a strictly higher value. Let $a_i := \inf\{\theta_i \mid (\theta_i, 0_{-i}) \in \mathcal{O}_S\}$, $b_i := \sup\{\theta_i \mid (\theta_i, 0_{-i}) \in \mathcal{O}_S\}$ and define $x^{(1)}(\theta_1, \theta_{-1}) := \frac{1}{F(b_1) - F(a_1)} \int_{a_1}^{b_1} \hat{x}(\beta, \theta_{-1}) dF(\beta)$. This function is constant in θ_1, feasible for the above problem given that $\hat{x}$ is feasible and, by Chebyshev's

inequality, for all θ_{-1},

$$\int_{a_1}^{b_1} \psi(\theta_1, \theta_{-1})\hat{x}(\theta_1, \theta_{-1})dF(\theta_1)$$

$$\leq \int_{a_1}^{b_1} \psi(\theta_1, \theta_{-1})dF(\theta_1)\frac{1}{F(b_1) - F(a_1)} \int_{a_1}^{b_1} \hat{x}(\theta_1, \theta_{-1})dF(\theta_1)$$

$$= \int_{a_1}^{b_1} \psi(\theta_1, \theta_{-1})x^{(1)}(\theta_1, \theta_{-1})dF(\theta_1).$$

Since this inequality holds point-wise, we also have

$$\int_{\mathcal{O}_S} \psi(\theta)\hat{x}(\theta)dF^n(\theta) \leq \int_{\mathcal{O}_S} \psi(\theta)x^{(1)}(\theta)dF^n(\theta).$$

Iteratively defining $x^{(j)}(\theta_j, \theta_{-j}) = \frac{1}{F(b_j)-F(a_j)} \int_{a_j}^{b_j} x^{(j-1)}(\beta, \theta_{-j})dF(\beta)$ for $j = 2, \ldots, n$ we get a function $x^{(n)}(\theta)$ that is constant in θ. Repeatedly applying Chebyshev's inequality along every dimension, we get

$$\int_{\mathcal{O}_S} \psi(\theta)\hat{x}(\theta)dF^n(\theta) \leq \int_{\mathcal{O}_S} \psi(\theta)x^{(n)}(\theta)dF^n(\theta).$$

Since the objective function is linear in x, the constant function $x^{(n)}$ is weakly dominated by either $x^* \equiv 1$ or $x^* \equiv 0$, contradicting the initial claim.

PROOF OF THEOREM 1:

Lemma 2 and Lemma 3 together imply that for any anonymous mechanism (x, t) it holds that

$$U(x, t) = \int_{\Theta^n} \left[\sum_{i \in N} \psi(\theta_i) \right] x(\theta)dF^n(\theta),$$

where ψ is defined in (A.1). $\sum_{i \in N} \psi(\theta_i)$ is non-increasing by the assumption of increasing hazard rates and Lemma 4 therefore implies that the optimal decision rule is constant and equal to 0 or 1 within each orthant. This implies that the optimal rule depends only on the sign of the reports.

Ex-ante symmetry of the agents implies that the solution to this problem is anonymous. Monotonicity of incentive compatible rules then implies that the optimal rule accepts the proposal if and only if the number of agents with positive types is above some threshold. The fact that the decision rule depends only on the signs of the reports implies moreover by Lemma 1 that $t_i(\theta) = 0$ for all i, θ.

Hence, it remains to determine the optimal cutoff for qualified majority voting.

Let k solve

$$k\mathbb{E}[\theta_i \mid \theta_i \geq 0] + (n - k)\mathbb{E}[\theta_i \mid \theta_i \leq 0] = 0.$$

Then the expected aggregate valuation, given that $k' < k$ agents are in favor of the proposal, is negative. Therefore, it is optimal to accept the proposal if and only if at least $\lceil k \rceil$ agents have a positive valuation.