Dynamic Games:
Numerical Methods and Applications

Mini Course, Tel Aviv University, Day 2

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Agenda

- Discussion of problem set.
- Application: Capacity accumulation.
- Application: Advertising dynamics.
- Markov-perfect industry dynamics.
- Existence, purification, and multiplicity of equilibrium.
- Application: Quality ladder model with entry/exit.

Quality Ladder Model without Entry/Exit

- Pakes, A. & McGuire, P. (1994) "Computing Markov-Perfect Nash Equilibria: Numerical Implications of a Dynamic Differentiated Product Model."
- Borkovsky, R., Doraszelski, U. & Kryukov, Y. (2010) "A User's Guide to Solving Dynamic Stochastic Games Using the Homotopy Continuation Method."
- Discrete time, infinite horizon.
- Two firms with potentially different product qualities

$$\omega = (\omega_1, \omega_2) \in \{1, \ldots, L\}^2 = \Omega.$$

- In each period, the timing is as follows:
  - Firms choose investments in quality improvements.
  - Product market competition takes place.
  - Investment outcomes and depreciation shocks are realized.

Product Market Competition

- Firm n's demand is

$$D_n(p_1, p_2; \omega) = M \frac{\exp\left(g(\omega_n) - p_n\right)}{1 + \sum_{k=1}^{2} \exp\left(g(\omega_k) - p_k\right)},$$

  where $M > 0$ is market size and

$$g(\omega_n) = \begin{cases} 3\omega_n - 4 & \text{if} \quad \omega_n \leq 5, \\ 12 + \ln\left(2 - \exp\left(16 - 3\omega_n\right)\right) & \text{if} \quad \omega_n > 5 \end{cases}$$

  maps product quality into consumers' valuations.

- Firm n solves

$$\max_{p_n \geq 0} D_n(p_1, p_2; \omega)(p_n - c),$$

  where c is marginal cost of production.

- FOC:

$$0 = 1 - \frac{1 + \exp\left(g(\omega_{-n}) - p_{-n}\right)}{1 + \exp\left(g(\omega_n) - p_n\right) + \exp\left(g(\omega_{-n}) - p_{-n}\right)}(p_n - c), \quad n \neq -n.$$

- Compute Nash equilibrium $(p_1(\omega), p_2(\omega))$ by solving system of FOCs.

- Firm n's profit is

$$\pi_n(\omega) = D_n(p_1(\omega), p_2(\omega); \omega)(p_n(\omega) - c).$$

Investment Dynamics

- Let $x_n \geq 0$ be firm n's investment in quality improvements.

- Law of motion:
  - Successful investment has probability $\frac{\alpha x_n}{1+\alpha x_n}$.
  - Depreciation shock has probability δ.

- Transition probability: If $\omega_n \in \{2, \ldots, L-1\}$, then

$$\Pr(\omega_n' | \omega_n, x_n) = \begin{cases} \frac{(1-\delta)\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = \omega_n + 1, \\ \frac{1-\delta+\delta\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = \omega_n, \\ \frac{\delta}{1+\alpha x_n} & \text{if} \quad \omega_n' = \omega_n - 1. \end{cases}$$

  If $\omega_n \in \{1, L\}$, then

$$\Pr(\omega_n' | 1, x_n) = \begin{cases} \frac{(1-\delta)\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = 2, \\ \frac{1+\delta\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = 1, \end{cases}$$

$$\Pr(\omega_n' | L, x_n) = \begin{cases} \frac{1-\delta+\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = L, \\ \frac{\delta}{1+\alpha x_n} & \text{if} \quad \omega_n' = L - 1. \end{cases}$$

Bellman Equation

- Let $V_n(\omega)$ denote the expected NPV to firm n if the current state is ω.
- Firm n's Bellman equation is

$$V_n(\omega) = \max_{x_n \geq 0} \pi_n(\omega) - x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, x_n),$$

  where

  - the expectation (with respect to its rival's successor state) of firm n's continuation value in state ω_n' is

  $$W_n(\omega_n'; \omega_{-n}, x_{-n}(\omega)) = \sum_{\omega_{-n}'=1}^{L} V_n(\omega') \Pr(\omega_{-n}' | \omega_{-n}, x_{-n}(\omega));$$

  - $x_{-n}(\omega)$ is the rival's investment strategy;
  - $\beta \in [0, 1)$ is the discount factor.

Investment Strategy

- Firm n's investment strategy is

$$x_n(\omega) = \arg\max_{x_n \geq 0} \pi_n(\omega) - x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n') \Pr(\omega_n' | \omega_n, x_n),$$

  where $W_n(\omega_n')$ is shorthand for $W_n(\omega_n'; \omega_{-n}, x_{-n}(\omega))$.

- If $\omega_n \in \{2, \dots, L-1\}$, then

$$x_n(\omega) = \frac{-1 + \sqrt{\max\{1, \beta\alpha\left((1-\delta)(W_n(\omega_n+1) - W_n(\omega_n)) + \delta(W_n(\omega_n) - W_n(\omega_n-1))\right)\}}}{\alpha}.$$

  If $\omega_n \in \{1, L\}$, then

$$x_n(\omega) = \frac{-1 + \sqrt{\max\{1, \beta\alpha(1-\delta)\left(W_n(2) - W_n(1)\right)\}}}{\alpha},$$

$$x_n(\omega) = \frac{-1 + \sqrt{\max\{1, \beta\alpha\delta\left(W_n(L) - W_n(L-1)\right)\}}}{\alpha}.$$

Equilibrium

- Profits from product market competition are symmetric:

$$\pi_1(\omega_1, \omega_2) = \pi_2(\omega_2, \omega_1).$$

  The remaining primitives are also symmetric.

- Symmetric Markov perfect equilibrium (MPE):
  - Value function $V_1(\omega_1, \omega_2) = V(\omega_1, \omega_2)$ and $V_2(\omega_1, \omega_2) = V(\omega_2, \omega_1)$.
  - Policy function $x_1(\omega_1, \omega_2) = x(\omega_1, \omega_2)$ and $x_2(\omega_1, \omega_2) = x(\omega_2, \omega_1)$.

- Existence in pure strategies is guaranteed (Doraszelski & Satterthwaite 2010), uniqueness is not.

- The goal is to compute the value and policy functions (or, more precisely, $L \times L$ matrices) $\mathbf{V}$ and $\mathbf{x}$.

Computation: Pakes & McGuire (1994) Algorithm

1. Make initial guesses $\mathbf{V}^0$ and $\mathbf{x}^0$, choose a stopping criterion $\epsilon > 0$, and initialize the iteration counter to $k = 1$.

2. For all states $\omega \in \Omega$ compute

$$x^{k+1}(\omega) = \arg\max_{x_1 \geq 0} \pi_1(\omega) - x_1 + \beta \sum_{\omega_1'=1}^{L} W^k(\omega_1')\Pr(\omega_1'|\omega_1, x_1)$$

   and

$$V^{k+1}(\omega) = \pi_1(\omega) - x^{k+1}(\omega) + \beta \sum_{\omega_1'=1}^{L} W^k(\omega_1')\Pr(\omega_1'|\omega_1, x^{k+1}(\omega)),$$

   where

$$W^k(\omega_1') = \sum_{\omega_2'=1}^{L} V^k(\omega')\Pr(\omega_2'|\omega_2, x^k(\omega_2, \omega_1)).$$

3. If

$$\max_{\omega\in\Omega}\left|\frac{V^{k+1}(\omega) - V^k(\omega)}{1 + |V^{k+1}(\omega)|}\right| < \epsilon \quad \wedge \quad \max_{\omega\in\Omega}\left|\frac{x^{k+1}(\omega) - x^k(\omega)}{1 + |x^{k+1}(\omega)|}\right| < \epsilon$$

   then stop; else increment the iteration counter k by one and go to step 2.

Application to Capacity Accumulation

- Besanko, D. & Doraszelski, U. (2004) "Capacity Dynamics and Endogenous Asymmetries in Firm Size."
- Substantial and persistent differences in firm sizes despite idiosyncratic shocks (Gort 1963, Mueller 1986, McGahan & Porter 1997).
- Size differences can arise endogenously in asymmetric equilibria of two- or three-stage models of capacity choice (Saloner 1987, Maggi 1996, Reynolds & Wilson 2000).
- But: What happens if firms are subject to idiosyncratic shocks? What about feedback effects?
- Dynamic models of capacity accumulation:
  - Steady-state analysis (Spence 1979, Fudenberg & Tirole 1983).
  - Linear-quadratic games (Hanig 1985, Reynolds 1987, 1991, Dockner 1992).

Relationship to Quality Ladder Model

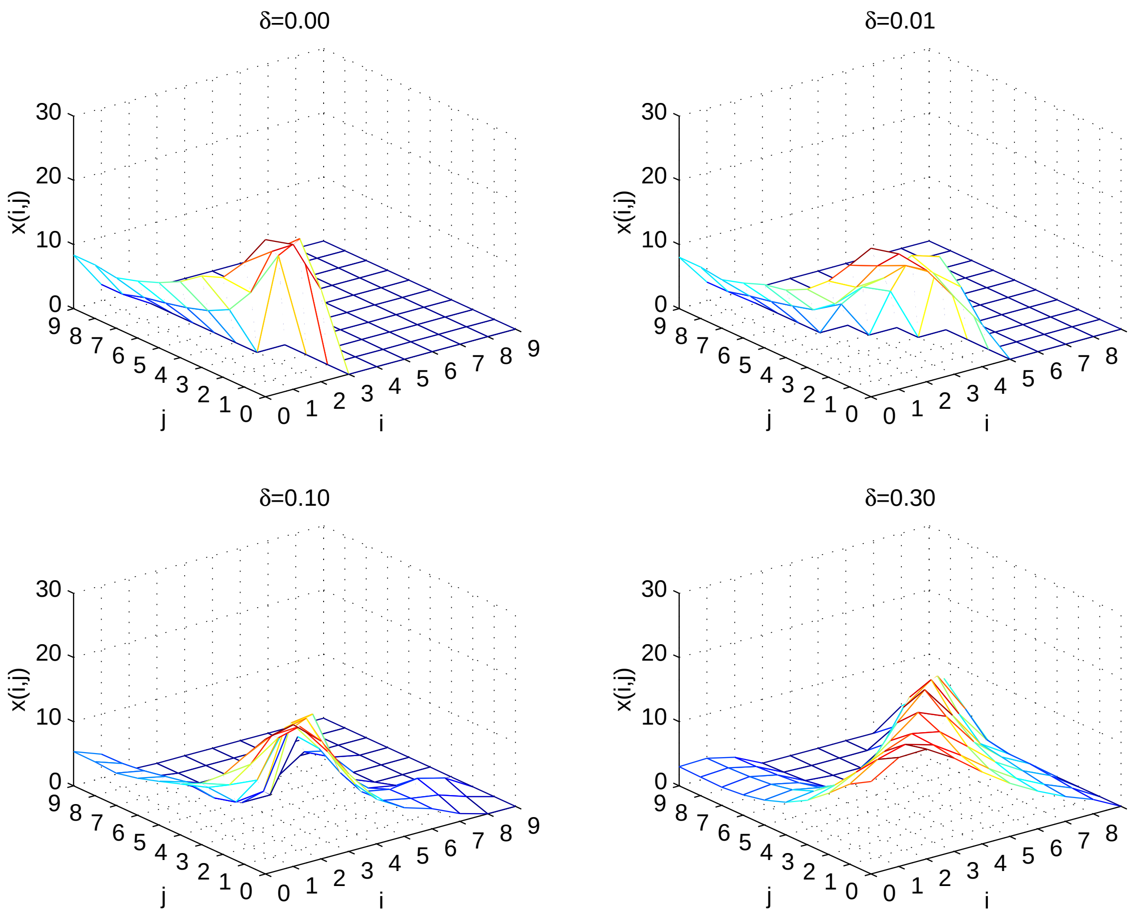
- State variables $\omega = (\omega_1, \omega_2)$ are capacities of firms 1 and 2.
- Firms invest in capacity. Capacity may depreciate.
- Product market competition:
  - Quantity competition subject to capacity constraints.
  - Price competition subject to capacity constraints.

Substantial and Persistent Differences in Firm Sizes

	quantity competition	**price competition**
irreversible investment ($\delta = 0$)	symmetric firms	slightly asymmetric firms
reversible investment ($\delta > 0$)	symmetric firms	hugely asymmetric firms

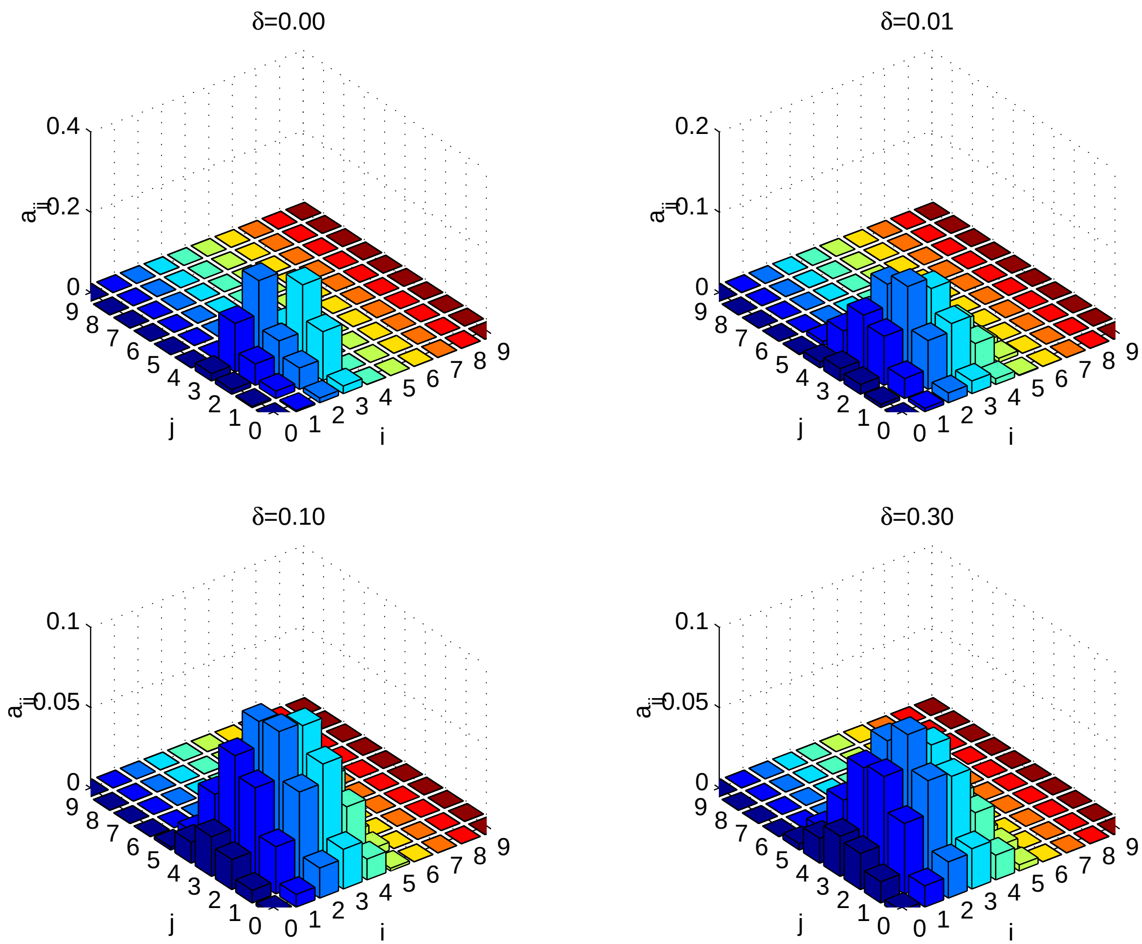
Investment Reversibility and Preemption Races

- Policy function $x(i,j)$. Price competition with $\delta \in \{0, 0.01, 0.1, 0.3\}$.



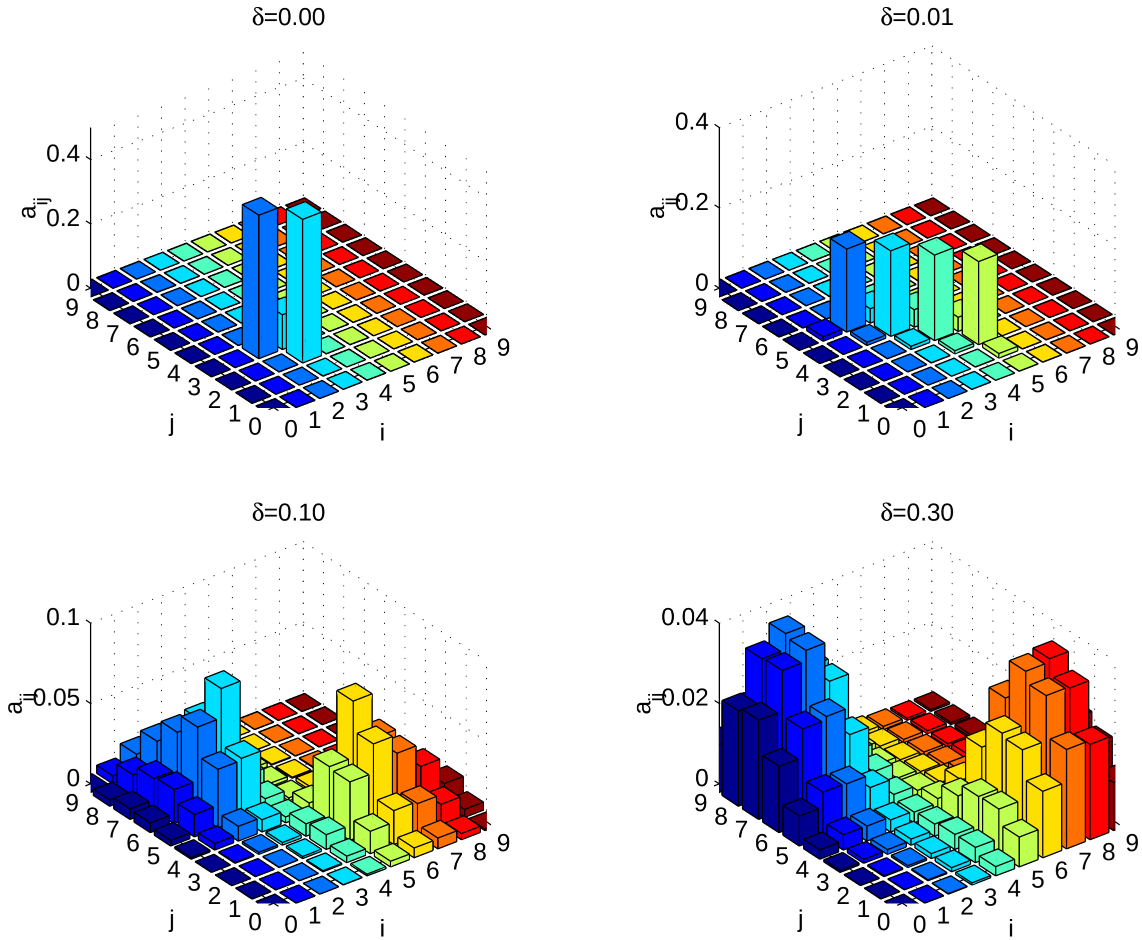
Investment Reversibility and Preemption Races

- Transient distribution after $T = 5$ with $i_0 = j_0 = 1$. Price competition with $\delta \in \{0, 0.01, 0.1, 0.3\}$.



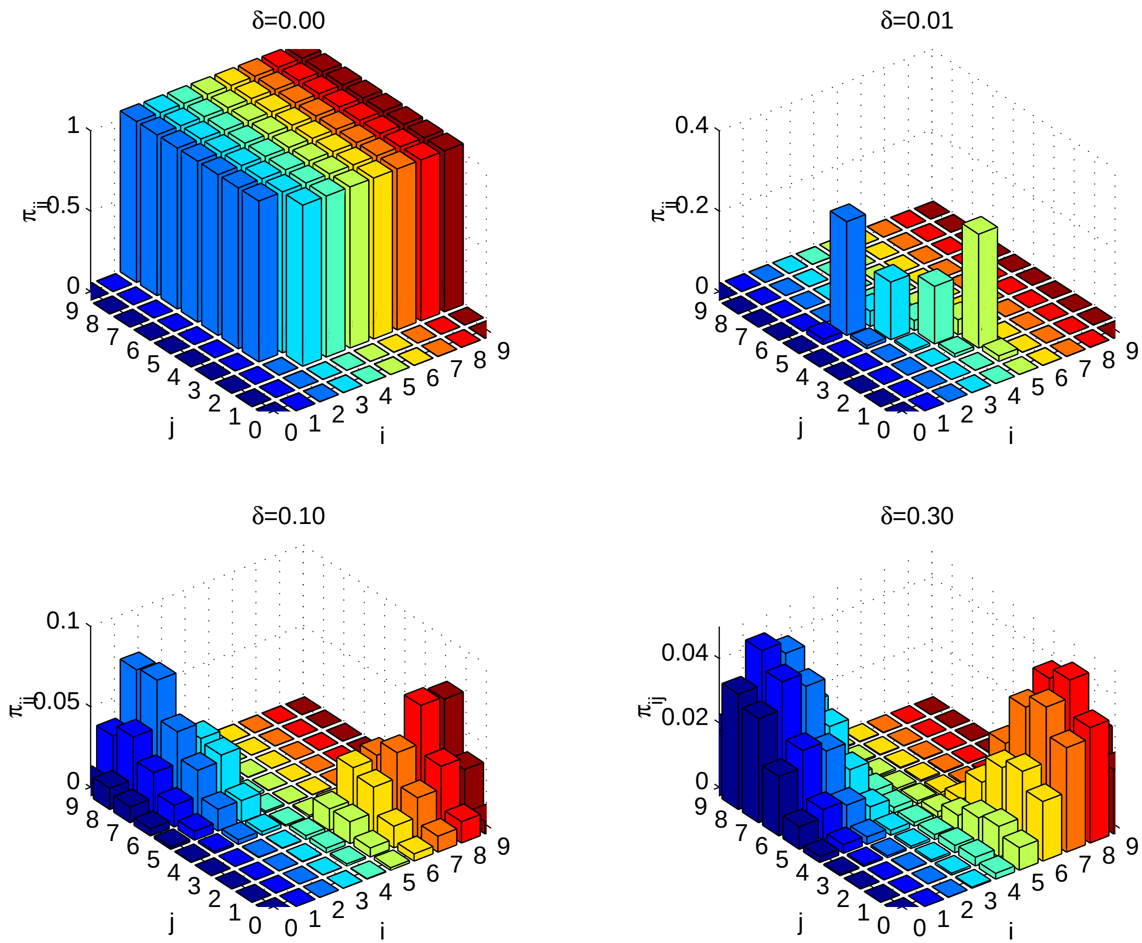
Investment Reversibility and Preemption Races

- Transient distribution after $T = 25$ with $i_0 = j_0 = 1$. Price competition with $\delta \in \{0, 0.01, 0.1, 0.3\}$.



Investment Reversibility and Preemption Races

- Limiting distribution. Price competition with $\delta \in \{0, 0.01, 0.1, 0.3\}$.



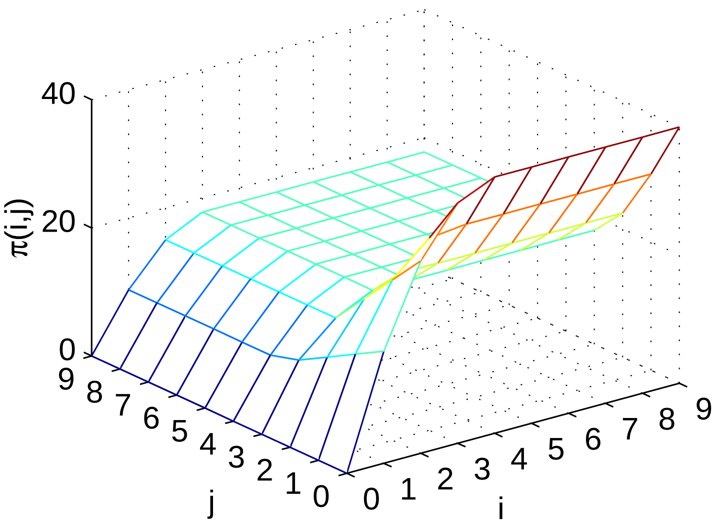
Investment Reversibility and Preemption Races

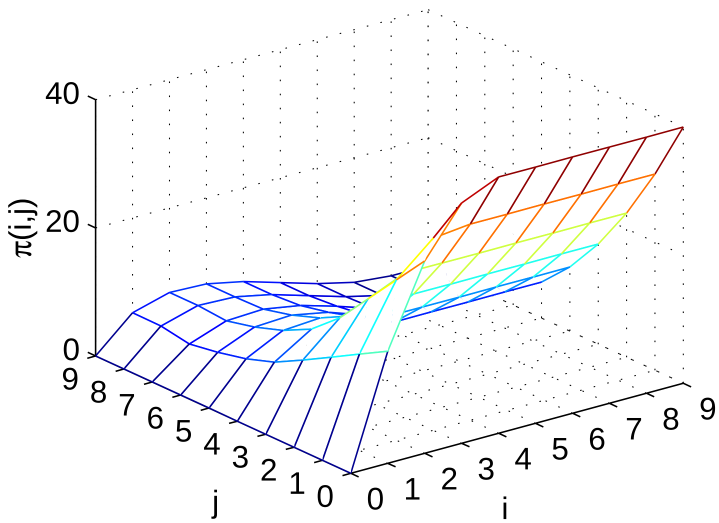
- “An open issue (...) is the behavior of investment in the industry when capital depreciates. Intuition suggests that capital ought to lose some of its commitment value and that the steady-state levels of capital should be less sensitive to the initial head start of one of the firms.” (Tirole 1988, p. 345)

- This paper: Investment reversibility may make preemption races *more* attractive.

Investment Reversibility and Preemption Races

- What is the main difference between the two modes of product market competition?
  - Capacity-constrained quantity competition: a firm's profit *plateaus* in own capacity.
  - Capacity-constrained price competition: a firm's profit *peaks* in own capacity (provided rival has sufficient capacity).





Investment Reversibility and Preemption Races

- Under price competition, it is in the self-interest of a not-too-small firm to withdraw from the race once its rival has gained a size advantage over it.

- By building up its capacity, a firm hopes to gain an initial edge over its rival and to decide the race in its favor.

- A firm anticipates that once it gains an edge over its rival, its rival will withdraw capacity.

- It is easier to withdraw capacity if the rate of depreciation is high. Conversely, it is impossible to withdraw capacity if the rate of depreciation is zero.

Application to Advertising Dynamics

- Doraszelski, U. & Markovich, S. (2007) "Advertising Dynamics and Competitive Advantage."

- Can advertising lead to a sustainable competitive advantage?

- Existing *static* models of advertising competition (Butters 1977, Grossman & Shapiro 1984, Boyer & Moreaux 1999) cannot address this question.

- Existing *dynamic* models of advertising competition (Friedman 1983, Fershtman 1984, Cellini & Lambertini 2003) say no (globally stable symmetric steady state).

- This paper: Yes!

Goodwill and Awareness Advertising

- Consumer m's problem is to choose among the products in his choice set C_m such that

$$\max_{n \in C_m} \left(v_n - p_n + \epsilon_{mn}\right).$$

- Goodwill advertising influences the utility that consumers derive from the product.
  - Persuasive advertising (Dixit & Norman 1978).
  - Complementary advertising (Stigler & Becker 1977, Becker & Murphy 1993).
- Awareness advertising influences the share of consumers who are aware of the product.
  - Informative advertising (Stigler 1961, Butters 1977, Grossman & Shapiro 1984).

Relationship to Quality Ladder Model

- Goodwill advertising: State variables $\mathbf{v} = (v_1, v_2)$ are perceived qualities of firms 1 and 2.

- Awareness advertising: State variables $\mathbf{s} = (s_1, s_2)$ are shares of consumers who are aware of firms 1 and 2.

- Firms invest in advertising. Goodwill/awareness may depreciate.

- Product market competition: Price competition with differentiated products.

Sustainable Competitive Advantage: Goodwill Advertising

small market/ expensive advertising	**large market/ cheap advertising**
extremely asymmetric firms	symmetric firms

Goodwill Advertising and Cost/Benefit Considerations

- Marginal benefit of advertising is determined by

$$\pi(v_1 + \Delta, v_2) - \pi(v_1, v_2)$$

  and is proportional to market size.

- In a small market, the marginal benefit is small.

- Marginal benefit is decreasing in rival's goodwill $\rightarrow$ large firm can deter small firm.

- Marginal benefit is increasing in firm's goodwill $\rightarrow$ large firm cannot deter medium or large firm.

Sustainable Competitive Advantage: Awareness Advertising

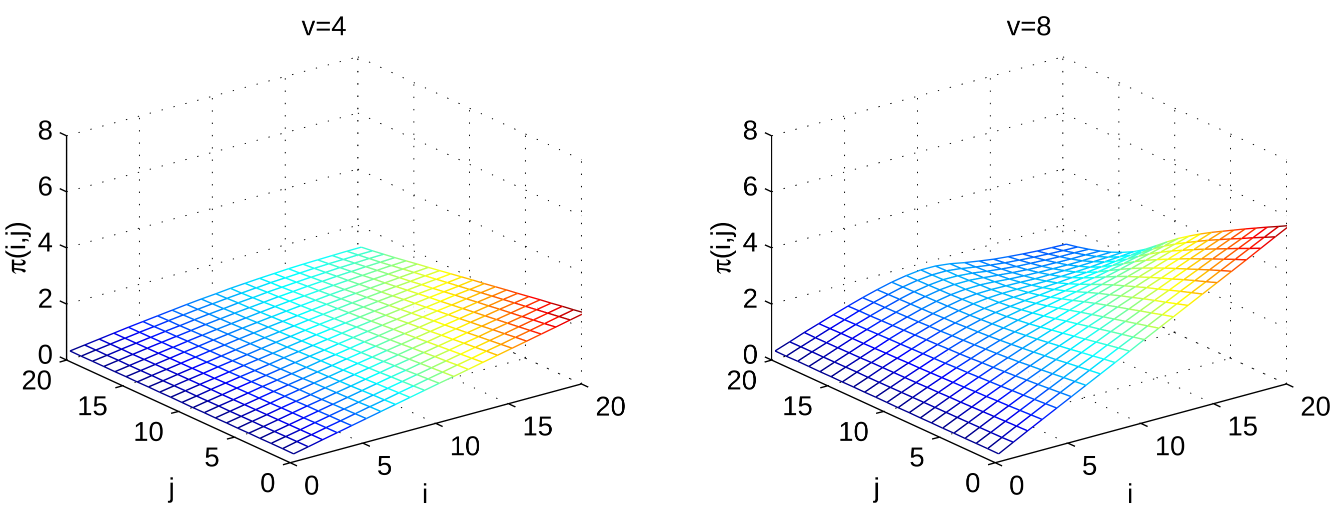
low perceived quality	**high perceived quality**
symmetric firms	asymmetric firms

Awareness Advertising and Product Market Competition

- What is the main difference between low and high perceived quality?

  – Low perceived quality: a firm's profit *increases* in own awareness.

  – High perceived quality: a firm's profit first increases then *decreases* in own awareness (provided rival has sufficient awareness).



Awareness Advertising and Product Market Competition

- Perceived quality of firms' products and intensity of competition.

- Captive segment vs competitive segment: The probability of buying from firm 1 is

$$D_1(p_1, p_2; s_1, s_2)$$

$$= s_1(1 - s_2) \underbrace{\frac{\exp(v - p_1)}{1 + \exp(v - p_1)}}_{\text{captive segment}} + s_1 s_2 \underbrace{\frac{\exp(v - p_1)}{1 + \exp(v - p_1) + \exp(v - p_2)}}_{\text{competitive segment}}.$$

- If $s_2 = 0$ ($s_2 = 1$), firm 1 set its monopolistic (duopolistic) price.

- More generally, $s_2 \uparrow \rightarrow p_1^* \downarrow \rightarrow p_2^* \downarrow$.

Awareness Advertising and Product Market Competition

- Strategic advantage is grounded in product market competition.

- With high perceived quality, a medium firm is better off staying put $\rightarrow$ large firm can deter medium firm.

- With high perceived quality, a small firm is better off trying to grow $\rightarrow$ large firm cannot deter small firm.

- Strategic advantage is independent of cost/benefit considerations.

Markov-Perfect Industry Dynamics

- Ericson, R. & Pakes, A. (1995) "Markov-Perfect Industry Dynamics: A Framework for Empirical Work."
- EP model tracks evolution of oligopolistic industries.
- Special case of dynamic game:
  - Entry, exit, and investment decisions.
  - Product market competition.
- Captures key findings of empirical literature on industry evolution:
  - Entry and exit occur simultaneously.
  - Heterogeneity among firms evolves endogenously and persists.

Applications in IO and Other Fields

- Advertising (Doraszelski & Markovich 2007).
- Capacity accumulation (Besanko & Doraszelski 2004, Chen 2009, Ryan 2012, Besanko, Doraszelski, Lu & Satterthwaite 2010a, 2010b, Wilson 2012).
- Collusion (Fershtman & Pakes 2000, 2005, de Roos 2004).
- Competitive convergence (Langohr 2003).
- Consumer learning (Ching 2010).
- Corporate reputation (Abito, Besanko & Diermeier 2012).
- Learning by doing (Benkard 2004, Besanko, Doraszelski, Kryukov & Satterthwaite 2010, Besanko, Doraszelski & Kryukov 2014, Besanko, Doraszelski & Kryukov 2016).
- Mergers (Berry & Pakes 1993, Gowrisankaran 1999, Mermelstein, Nocke, Satterthwaite & Whinston 2013).
- Network effects (Jenkins, Liu, Matzkin & McFadden 2004, Markovich 2004, Markovich & Moenius 2005, Chen, Doraszelski & Harrington 2009).
- Productivity growth (Laincz 2005).
- R&D (Gowrisankaran & Town 1997, Auerswald 2001, Song 2011).
- Switching costs (Chen 2011).
- Technology adoption (Schivardi & Schneider 2005).
- International trade (Erdem & Tybout 2003).
- Finance (Goettler, Parlour & Rajan 2004).

Connections to Operations Research and Applied Math Literatures

- Discrete-time games go back to Shapley (1953), continuous-time games to Isaacs (1954).
- Markov perfect equilibrium (Maskin & Tirole 2001) "rediscovers" feedback Nash equilibrium.
- Lots of existence proofs (Sobel 1971, Federgruen 1976, Whitt 1980).
- Less on algorithms.
- Not everything is useful for economics (zero-sum games, average-payoff games).
- Good textbooks: Filar & Vrieze (1997), Basar & Olsder (1999).

Connections to Economics Literature

- EP model combines literature on long-run industry equilibrium (Jovanovic 1982, Hopenhayn 1992, Melitz 2003) with game theory (Tirole 1988, Fudenberg & Tirole 1991).

- EP model builds on analytically tractable special cases of dynamic games:

  - exponential games (Loury 1979, Lee & Wilde 1980, Reinganum 1982).

  - linear-quadratic games (Friedman 1983, Fershtman 1984, Reynolds 1987, 1991, Dockner 1992).

Existence, Purification, and Multiplicity of Equilibrium

- Doraszelski, U. & Satterthwaite, M. (2010) "Computable Markov-Perfect Industry Dynamics."

- Questions:
  - Does a MPE exist in the EP model?
  - Is the MPE computationally tractable?
    * Pure strategies.
    * Symmetric and anonymous (exchangeable).
  - Is the MPE unique?

- Answers:
  - In the EP model a symmetric and anonymous MPE in pure strategies always exists under reasonable conditions.
  - The MPE is not necessarily unique.

Three Difficulties

- Randomization over discrete actions (entry/exit):
  - Introduce randomly drawn, privately-known setup costs/scrap values $\rightarrow$ the game of incomplete information has a MPE in cutoff entry/exit strategies.

- Randomization over continuous actions (investment):
  - Provide conditions on the model's primitives (UIC admissibility) such that a firm's optimal investment level is always unique $\rightarrow$ the MPE is in pure investment strategies.
  - Recent generalization: Escobar, J. (2013) "Equilibrium Analysis of Dynamic Models of Imperfect Competition."

- Symmetry and anonymity.
  - Provide conditions on the model's primitives $\rightarrow$ the MPE is symmetric and anonymous.

Quality Ladder Model with Entry/Exit

- Pakes, A. & McGuire, P. (1994) "Computing Markov-Perfect Nash Equilibria: Numerical Implications of a Dynamic Differentiated Product Model."

- Borkovsky, R., Doraszelski, U. & Kryukov, Y. (2012) "A Dynamic Quality Ladder Model with Entry and Exit: Exploring the Equilibrium Correspondence Using the Homotopy Method."

- Incumbent firms (i.e., active firms) and potential entrants (i.e., inactive firms).

- Two firms that can be either a potential entrant or an incumbent firm with potentially different product qualities

$$\omega = (\omega_1, \omega_2) \in \{ \underbrace{1, \ldots, L}_{\text{active firm}}, \quad \underbrace{L+1}_{\text{inactive firm}} \quad \}^2 = \Omega.$$

- Exit is a transition from state $\omega_n \neq L + 1$ to state $\omega'_n = L + 1$.

- Entry is a transition from state $\omega_n = L + 1$ to state $\omega'_n = \omega^e \neq L + 1$, where ω^e is an exogenously given initial product quality.

Quality Ladder Model with Entry/Exit

- Let $\xi_n(\omega) \in [0,1]$ be firm n's probability of remaining in (if $\omega_n \neq L+1$) or entering into (if $\omega_n = L+1$) the industry.

- Transition probability: If $\omega_n \in \{2, \ldots, L-1\}$, then

$$\Pr(\omega_n' | \omega_n, \xi_n, x_n) = \begin{cases} \xi_n \frac{(1-\delta)\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = \omega_n + 1, \\ \xi_n \frac{1-\delta+\delta\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = \omega_n, \\ \xi_n \frac{\delta}{1+\alpha x_n} & \text{if} \quad \omega_n' = \omega_n - 1, \\ 1 - \xi_n & \text{if} \quad \omega_n' = L+1, \end{cases}$$

etc. If $\omega_n = L+1$, then

$$\Pr(\omega_n' | \omega_n, \xi_n) = \begin{cases} \xi_n & \text{if} \quad \omega_n' = \omega^e, \\ 1 - \xi_n & \text{if} \quad \omega_n' = L+1. \end{cases}$$

Quality Ladder Model with Entry/Exit

- Firm n is assigned a random scrap value $\phi_n \sim F$ (if $\omega_n \neq L+1$) or a random setup cost $\phi_n^e \sim F^e$ (if $\omega_n = L+1$).
  - Scrap values/setup costs are private information.
  - Scrap values/setup costs are independent across firms and periods.
- Because scrap values and setup costs are private to a firm, its rivals perceive the firm *as if* it is mixing.
- In each period the timing is as follows:
  - Incumbent firms learn their scrap value and decide on exit and investment. Potential entrants learn their setup cost and decide on entry and investment.
  - Incumbent firms compete in the product market.
  - Exit and entry decisions are implemented.
  - The investment decisions of the remaining incumbents and new entrants are carried out and their uncertain outcomes are realized.

Incumbent Firm

- Bellman equation without entry/exit:

$$V_n(\omega) = \max_{x_n \geq 0} \pi_n(\omega) - x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, x_{-n}(\omega)) \mathrm{Pr}(\omega_n' | \omega_n, x_n).$$

- Bellman equation with entry/exit:

$$V_n(\omega) = \max_{\xi_n \in [0,1], x_n \geq 0} \pi_n(\omega) + (1 - \xi_n) \mathrm{E}\left\{\phi_n | \phi_n \geq F^{-1}(\xi_n)\right\}$$

$$+\xi_n \left\{ -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \mathrm{Pr}(\omega_n' | \omega_n, x_n, \xi_n = 1) \right\},$$

where

$$(1 - \xi_n) \mathrm{E}\left\{\phi_n | \phi_n \geq F^{-1}(\xi_n)\right\} = \int_{\phi_n \geq F^{-1}(\xi_n)} \phi_n dF(\phi_n).$$

- An optimizing incumbent cares about the scrap value conditional on receiving it.

- Optimality condition:

$$\xi_n(\omega) = F\left(-x_n(\omega) + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \mathrm{Pr}(\omega_n' | \omega_n, x_n(\omega), \xi_n = 1) \right).$$

Incumbent Firm: Derivation of Bellman Equation

- Let $\chi_n(\omega,\phi_n) \in \{0,1\}$ be firm n's decision of remaining in (if $\omega_n \neq L+1$) or entering into (if $\omega_n = L+1$) the industry.

- Let $\xi_n(\omega) = \int \chi_n(\omega,\phi_n)dF(\phi_n)$ be firm n's probability of remaining in (if $\omega_n \neq L+1$) or entering into (if $\omega_n = L+1$) the industry (as perceived by its rivals).

- Let $V_n(\omega,\phi_n)$ be the value function of incumbent firm n *after* it observes its scrap value.

- Bellman equation:

$$V_n(\omega,\phi_n) = \max_{\chi_n \in \{0,1\}, x_n \geq 0} \pi_n(\omega) + (1-\chi_n)\phi_n$$

$$+\chi_n \left\{ -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, x_n, \xi_n = 1) \right\},$$

- The problem of the incumbent can be broken up into two parts:
  - The incumbent chooses its investment conditional on remaining in the industry $\rightarrow$ the optimal investment choice is independent of the firm's scrap value.
  - Given its investment choice, the incumbent decides whether or not to remain in the industry.

Incumbent Firm: Derivation of Bellman Equation

- Optimal decision has reservation property:

$$\chi_n(\omega, \phi_n) = \begin{cases} 1 & \text{if} \quad \phi_n \leq \bar{\phi}_n(\omega), \\ 0 & \text{if} \quad \phi_n \geq \bar{\phi}_n(\omega), \end{cases}$$

  where

$$\bar{\phi}_n(\omega) = \max_{x_n \geq 0} -x_n + \beta \sum_{\omega'_n=1}^{L} W_n(\omega'_n; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega'_n | \omega_n, x_n, \xi_n = 1)$$

  denotes the cutoff scrap value.

- Restrict attention to decision rules of the form $1[\phi_n < \bar{\phi}_n(\omega)]$.

- Instead of the cutoff $\bar{\phi}_n(\omega)$, represent these rules with the induced probability $\xi_n(\omega)$:

$$\xi_n(\omega) = \int \chi(\omega, \phi_n) dF(\phi_n) = \int 1[\phi_n < \bar{\phi}_n(\omega)] dF(\phi_n) = F(\bar{\phi}_n(\omega))$$
$$\Leftrightarrow \bar{\phi}_n(\omega) = F^{-1}(\xi_n(\omega))$$

  provided F has positive density on its support.

Incumbent Firm: Derivation of Bellman Equation

- Imposing the reservation property on the Bellman equation yields

$$
\begin{aligned}
V_n(\omega, \phi_n) = & \max_{\chi_n \in \{0,1\}, x_n \geq 0} \pi_n(\omega) + (1 - \chi_n)\phi_n \\
& + \chi_n \left\{ -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, x_n, \xi_n = 1) \right\} \\
= & \max_{\xi_n \in [0,1], x_n \geq 0} \pi_n(\omega) + (1 - 1[\phi_n < F^{-1}(\xi_n)])\phi_n \\
& + 1[\phi_n < F^{-1}(\xi_n)] \left\{ -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, x_n, \xi_n = 1) \right\}.
\end{aligned}
$$

Incumbent Firm: Derivation of Bellman Equation

- Let $V_n(\omega) = \int V_n(\omega, \phi_n) dF(\phi_n)$ be the value function of incumbent firm n *before* it observes its scrap value.

- Integrating over ϕ_n on both sides of the Bellman equation yields

$$V_n(\omega) = \int \max_{\xi_n \in [0,1], x_n \geq 0} \pi_n(\omega) + (1 - 1[\phi_n < F^{-1}(\xi_n)])\phi_n$$

$$+1[\phi_n < F^{-1}(\xi_n)] \left\{ -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, x_n, \xi_n = 1) \right\} dF(\phi_n)$$

$$= \max_{\xi_n \in [0,1], x_n \geq 0} \pi_n(\omega) + \int_{\phi_n \geq F^{-1}(\xi_n)} \phi_n dF(\phi_n)$$

$$+\xi_n \left\{ -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, x_n, \xi_n = 1) \right\}.$$

Potential Entrant

- Potential entrants are short-lived.

- Bellman equation:

$$V_n(\omega) = \max_{\xi_n \in [0,1]} \xi_n \Bigg\{ - \mathsf{E}\left\{\phi_n^e | \phi_n^e \leq F^{e-1}(\xi_n)\right\}$$

$$+\beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, \xi_n = 1) \Bigg\},$$

where

$$\xi_n \mathsf{E}\left\{\phi_n^e | \phi_n^e \leq F^{e-1}(\xi_n)\right\} = \int_{\phi_n^e \leq F^{e-1}(\xi_n)} \phi_n^e dF^e(\phi_n^e).$$

- An optimizing entrant cares about the setup cost conditional on paying it.

- Optimality condition:

$$\xi_n(\omega) = F^e \left(\beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, \xi_n = 1) \right).$$