# Dynamic Games: <br> Numerical Methods and Applications 

Mini Course, Tel Aviv University, Day 3<br>Prof. Ulrich Doraszelski, University of Pennsylvania

## Agenda

- Discussion of problem set.
- Symmetry and anonymity.
- Multiple equilibria.
- Application: Learning-by-doing.


## Symmetry and Anonymity

- There are $N$ firms. The state of firm $n$ is $\omega_{n} \in\{1,2, \ldots, L\}$. The state space is $\Omega=\{1,2, \ldots, L\}^{N}$.
- Symmetry allows us to focus on the problem of firm 1. Formally,

$$
V_{n}\left(\omega_{1}, \ldots, \omega_{n-1}, \omega_{n}, \omega_{n+1}, \ldots, \omega_{N}\right)=V_{1}\left(\omega_{n}, \ldots, \omega_{n-1}, \omega_{1}, \omega_{n+1}, \ldots, \omega_{N}\right)
$$

for all $n$, and similarly for the policy function $X_{n}(\omega)$.

- Anonymity (exchangeability) says that firm 1 does not care about the identity of its competitors. Formally,

$$
V_{1}\left(\omega_{1}, \omega_{2}, \ldots, \omega_{k}, \ldots, \omega_{l}, \ldots, \omega_{N}\right)=V_{1}\left(\omega_{1}, \omega_{2}, \ldots, \omega_{l}, \ldots, \omega_{k}, \ldots, \omega_{N}\right)
$$

for all $k \geq 2$ and $l \geq 2$, and similarly for the policy function $X_{n}(\omega)$.

- Under suitable symmetry and anonymity assumptions on the model's primitives, a symmetric and anonymous equilibrium exists (Doraszelski \& Satterthwaite 2010).
- Greatly reduces computational burden.


## Symmetry and Anonymity

- Consider the quality ladder model without entry/exit.
- The profit functions are symmetric and anonymous, i.e.,

$$
\pi_{n}\left(\omega_{1}, \ldots, \omega_{n-1}, \omega_{n}, \omega_{n+1}, \ldots, \omega_{N}\right)=\pi_{1}\left(\omega_{n}, \ldots, \omega_{n-1}, \omega_{1}, \omega_{n+1}, \ldots, \omega_{N}\right)
$$

for all $n$ and

$$
\pi_{1}\left(\omega_{1}, \omega_{2}, \ldots, \omega_{k}, \ldots, \omega_{l}, \ldots, \omega_{N}\right)=\pi_{1}\left(\omega_{1}, \omega_{2}, \ldots, \omega_{l}, \ldots, \omega_{k}, \ldots, \omega_{N}\right)
$$

for all $k \geq 2$ and all $l \geq 2$.

- The transition function is anonymous, i.e.,

$$
\begin{gathered}
\operatorname{Pr}\left(\omega_{1}^{\prime}, \ldots, \omega_{k}^{\prime}, \ldots, \omega_{l}^{\prime}, \ldots, \omega_{N}^{\prime}, \omega_{1}, \ldots, \omega_{k}, \ldots, \omega_{l}, \ldots, \omega_{N}\right. \\
\left.x_{1}(\omega), \ldots, x_{k}(\omega), \ldots, x_{l}(\omega), \ldots, x_{N}(\omega)\right) \\
=\operatorname{Pr}\left(\omega_{1}^{\prime}, \ldots, \omega_{l}^{\prime}, \ldots, \omega_{k}^{\prime}, \ldots, \omega_{N}^{\prime}, \omega_{1}, \ldots, \omega_{l}, \ldots, \omega_{k}, \ldots, \omega_{N}\right. \\
\left.x_{1}(\omega), \ldots, x_{l}(\omega), \ldots, x_{k}(\omega), \ldots, x_{N}(\omega)\right)
\end{gathered}
$$

for all $k \geq 1$ and all $l \geq 1$.

## Multiple Equilibria

... we have experimented quite a bit with the core version of the algorithm, and we never found two sets of equilibrium policies for a given set of primitives (we frequently run the algorithm several times using different initial conditions or different orderings of points looking for other equilibria that might exist). We should emphasize here that the core version, and indeed most other versions that have been used, all use quite simple functional forms for the primitives of the problem, and multiplicity of equilibrium may well be more likely when more complicated functional forms are used. Of course, most applied work suffices with quite simple functional forms. (Pakes 2000, pp. 18-19)

## Multiple Equilibria: Quality Ladder Model without Entry/Exit



Number of equilibria in the Pakes \& McGuire (1994) quality ladder model without entry/exit.
Source: Borkovsky, Doraszelski \& Kryukov (2010).

## Multiple Equilibria: Quality Ladder Model with Entry/Exit



Number of equilibria in the Pakes \& McGuire (1994) quality ladder model with entry/exit. Source: Borkovsky, Doraszelski \& Kryukov (2012).

## Multiple Equilibria

...I should note that virtually all Markov Perfect Models have multiple equilibria... (anonymous referee, 2013)

## Multiple Equilibria in Estimation

Nested-fixed point algorithm:

- Gowrisankaran, G. \& Town, R. (1997) "Dynamic equilibrium in the hospital industry."

Two-step methods:

- Aguirregabiria, V. \& Mira, P. (2007) "Sequential Estimation of Dynamic Discrete Games."

Additional readings:

- Pesendorfer, M. \& Schmidt-Dengler, P. (2010) "Sequential Estimation of Dynamic Discrete Games: A Comment."
- Kasahara, H \& Shimotsu, K. (2012) "Sequential Estimation of Structural Models with a Fixed Point Constraint."
- Bajari, P., Benkard, L. \& Levin, J. (2007) "Estimating Dynamic Models of Imperfect Competition."
- Pakes, A., Ostrovsky, M. \& Berry, S. (2007) "Simple Estimators for the Parameters of Discrete Dynamic Games (with Entry/Exit Examples)."
- Pesendorfer, M. \& Schmidt-Dengler, P. (2008) "Asymptotic Least Squares Estimators for Dynamic Game."

MPEC method:

- Judd, K. \& Su, C. (2012) "Constrained Optimization Approaches to Estimation of Structural Models."


## Multiple Equilibria in Counterfactual Analysis

Out-of-equilibrium adjustment processes:

- Lee, R. \& Pakes, A. (2009) "Multiple Equilibria and Selection by Learning in an Applied Setting."
- Doraszelski, U. \& Escobar, J. (2010) "A Theory of ReguIar Markov Perfect Equilibria in Dynamic Stochastic Games: Genericity, Stability, and Purification."
- Aguirregabiria, V. (2012) "A Method for Implementing Counterfactual Experiments in Models with Multiple Equilibria."
- Doraszelski, U., Lewis, G. \& Pakes, A. (2015) "Just Starting Out: Learning and Equilibrium in a New Market."


## Learning-by-Doing

- Besanko, D., Doraszelski, U., Kryukov, S. \& Satterthwaite, M. (2010) "Learning-by-Doing, Organizational Forgetting, and Industry Dynamics."
- Question: Is organizational forgetting an antidote to market dominance?
- Incorporate organizational forgetting into the Cabral \& Riordan (1994) model of learning-by-doing.
- Dynamic competition with learning-by-doing and organizational forgetting is akin to racing down an upward-moving escalator.
- Organizational forgetting makes bidirectional movements through the state space possible. Thus, it is a source of...
- ...aggressive pricing behavior;
- . . . market dominance;
- ...multiple equilibria.
- Learning-by-doing and organizational forgetting are distinct economic forces.


## Learning-by-Doing

- Discrete time, infinite horizon.
- Two firms with potentially different stocks of know-how

$$
\omega=\left(\omega_{1}, \omega_{2}\right) \in\{1, \ldots, L\}^{2}=\Omega
$$

- In each period, the timing is as follows:
- Firms choose prices.
- One buyer enters the market and makes at most one purchase.
- Learning-by-doing and organizational forgetting occur and the firms' stocks of know-how change accordingly.
- Law of motion:
where
- $q_{n} \in\{0,1\}$ indicates whether firm $n$ makes a sale with

$$
\operatorname{Pr}\left(q_{n}=1\right)=D_{n}\left(p_{1}, p_{2}\right)=\frac{\exp \left(v-p_{n}\right)}{1+\sum_{k=1}^{2} \exp \left(v-p_{k}\right)}
$$

- $f_{n} \in\{0,1\}$ represents organizational forgetting with

$$
\operatorname{Pr}\left(f_{n}=1\right)=\Delta\left(\omega_{n}\right)=1-(1-\delta)^{\omega_{n}} .
$$

## Bellman Equation

- Let $V_{n}(\omega)$ denote the expected NPV to firm $n$ if the current state is $\omega$.
- Firm n's Bellman equation is

$$
V_{n}(\omega)=\max _{p_{n}} D_{n}\left(p_{n}, p_{-n}(\omega)\right)\left(p_{n}-c\left(\omega_{n}\right)\right)+\beta \sum_{k=0}^{2} D_{k}\left(p_{n}, p_{-n}(\omega)\right) W_{n k}(\omega),
$$

where

- $p_{-n}(\omega)$ is the price charged by the other firm;
- the marginal cost of production is

$$
c\left(\omega_{n}\right)=\left\{\begin{array}{ccc}
\kappa \omega_{n}^{\eta} & \text { if } & 1 \leq \omega_{n}<l, \\
\kappa l^{\eta} & \text { if } & l \leq \omega_{n} \leq L,
\end{array}\right.
$$

with $\eta=\log _{2} \rho$ for a progress ratio of $\rho$;
$-\beta \in(0,1)$ is the discount factor;

- $W_{n k}(\omega)$ is the expectation of firm $n$ 's value function conditional on buyer purchasing good $k \in\{0,1,2\}$ (good 0 is outside good).


## Bellman Equation

- Continuation values:

$$
\begin{aligned}
& W_{n 0}(\omega)=\sum_{\omega_{1}^{\prime}=1}^{L} \sum_{\omega_{2}^{\prime}=1}^{L} V_{n}\left(\omega^{\prime}\right) \operatorname{Pr}\left(\omega_{1}^{\prime} \mid \omega_{1}, q_{1}=0\right) \operatorname{Pr}\left(\omega_{2}^{\prime} \mid \omega_{2}, q_{2}=0\right) \\
& W_{n 1}(\omega)=\sum_{\omega_{1}^{\prime}=1}^{L} \sum_{\omega_{2}^{\prime}=1}^{L} V_{n}\left(\omega^{\prime}\right) \operatorname{Pr}\left(\omega_{1}^{\prime} \mid \omega_{1}, q_{1}=1\right) \operatorname{Pr}\left(\omega_{2}^{\prime} \mid \omega_{2}, q_{2}=0\right) \\
& W_{n 2}(\omega)=\sum_{\omega_{1}^{\prime}=1}^{L} \sum_{\omega_{2}^{\prime}=1}^{L} V_{n}\left(\omega^{\prime}\right) \operatorname{Pr}\left(\omega_{1}^{\prime} \mid \omega_{1}, q_{1}=0\right) \operatorname{Pr}\left(\omega_{2}^{\prime} \mid \omega_{2}, q_{2}=1\right)
\end{aligned}
$$

where

$$
\operatorname{Pr}\left(\omega_{n}^{\prime} \mid \omega_{n}, q_{n}\right)=\left\{\begin{array}{ccc}
1-\Delta\left(\omega_{n}\right) & \text { if } & \omega_{n}^{\prime}=\omega_{n}+q_{n} \\
\Delta\left(\omega_{n}\right) & \text { if } & \omega_{n}^{\prime}=\omega_{n}+q_{n}-1,
\end{array}\right.
$$

and $\operatorname{Pr}\left(L \mid L, q_{n}=1\right)=1$ and $\operatorname{Pr}\left(1 \mid 1, q_{n}=0\right)=1$.

## Pricing Strategy

- $p_{n}(\omega)$ is unique solution to FOC:

$$
\begin{gathered}
0=1-\left(1-D_{n}\left(p_{n}, p_{-n}(\omega)\right)\right)\left(p_{n}-c\left(\omega_{n}\right)\right)-\beta W_{n n}(\omega) \\
+\beta \sum_{k=0}^{2} D_{k}\left(p_{n}, p_{-n}(\omega)\right) W_{n k}(\omega) .
\end{gathered}
$$

- No closed-firm solution. Solve numerically.


## Equilibrium

- Primitives are symmetric.
- Symmetric Markov perfect equilibrium (MPE):
- Value function $V_{1}\left(\omega_{1}, \omega_{2}\right)=V\left(\omega_{1}, \omega_{2}\right)$ and $V_{2}\left(\omega_{1}, \omega_{2}\right)=V\left(\omega_{2}, \omega_{1}\right)$.
- Policy function $p_{1}\left(\omega_{1}, \omega_{2}\right)=p\left(\omega_{1}, \omega_{2}\right)$ and $p_{2}\left(\omega_{1}, \omega_{2}\right)=p\left(\omega_{2}, \omega_{1}\right)$.
- Existence in pure strategies is guaranteed (Doraszelski \& Satterthwaite 2010), uniqueness is not.
- The goal is to compute the value and policy functions (or, more precisely, $L \times L$ matrices) $\mathbf{V}$ and $\mathbf{p}$.


## Computation: Pakes \& McGuire (1994) Algorithm

1. Make initial guesses $\mathbf{V}^{0}$ and $\mathbf{p}^{0}$, choose a dampening factor $\lambda \in(0,1]$, choose a stopping criterion $\epsilon>0$, and initialize the iteration counter to $l=1$.
2. For all states $\omega \in \Omega$ compute

$$
\begin{aligned}
& p^{l+1}(\omega)=\arg \max _{p_{1}} D_{1}\left(p_{1}, p^{l}\left(\omega_{2}, \omega_{1}\right)\right)\left(p_{1}-c\left(\omega_{1}\right)\right) \\
&+\beta \sum_{k=0}^{2} D_{k}\left(p_{1}, p^{l}\left(\omega_{2}, \omega_{1}\right)\right) W_{k}^{l}(\omega)
\end{aligned}
$$

and

$$
\begin{aligned}
& V^{l+1}(\omega)=D_{1}\left(p^{l+1}(\omega), p^{l}\left(\omega_{2}, \omega_{1}\right)\right)\left(p^{l+1}(\omega)-c\left(\omega_{1}\right)\right) \\
&+\beta \sum_{k=0}^{2} D_{k}\left(p^{l+1}(\omega), p^{l}\left(\omega_{2}, \omega_{1}\right)\right) W_{k}^{l}(\omega)
\end{aligned}
$$

3. Dampening step: Assign

$$
\begin{aligned}
\mathbf{V}^{l+1} & \leftarrow \lambda \mathbf{V}^{l+1}+(1-\lambda) \mathbf{V}^{l} \\
\mathbf{p}^{l+1} & \leftarrow \lambda \mathbf{p}^{l+1}+(1-\lambda) \mathbf{p}^{l}
\end{aligned}
$$

4. If

$$
\max _{\omega \in \Omega}\left|\frac{V^{l+1}(\omega)-V^{l}(\omega)}{1+\left|V^{l+1}(\omega)\right|}\right|<\epsilon \quad \wedge \quad \max _{\omega \in \Omega}\left|\frac{p^{l+1}(\omega)-p^{l}(\omega)}{1+\left|p^{l+1}(\omega)\right|}\right|<\epsilon
$$

then stop; else increment the iteration counter $l$ by one and go to step 2.

## Categories of Equilibria

Flat Eqbm. without Well $(\rho=0.85, \delta=0)$
Flat Eqbm. with Well $(\rho=0.85, \delta=0.0275$ )

Categories of equilibria:

- flat without well;
- flat with well;
- trenchy;
- extra-trenchy.


Trenchy Eqbm. $\rho=0.85, \delta=0.0275$ )



Extra-trenchy Eqbm. ( $\rho=0.85, \delta=0.08$ )


Policy function $p^{*}\left(\omega_{1}, \omega_{2}\right)$. Marginal cost $c\left(\omega_{1}\right)$ (solid line in $\omega_{2}=30$-plane).

## Industry Dynamics: Flat Equilibrium with Well ( $\rho=0.85, \delta=0.03$ )



Transient distribution over states in periods 8, 16, and 32 given initial state (1, 1) and limiting distribution.

Industry Dynamics: Trenchy Equilibrium ( $\rho=0.85, \delta=0.03$ )


Transient distribution over states in periods 8, 16, and 32 given initial state (1, 1) and limiting distribution.

## Multiple Equilibria

Proposition 1 If organizational forgetting is either absent ( $\delta=0$ ) or certain ( $\delta=1$ ), then there is a unique equilibrium.

Result 1 If organizational forgetting is neither absent ( $\delta=0$ ) nor certain ( $\delta=1$ ), then there may be multiple equilibria.


## Organizational Forgetting and Multiple Equilibria

- What gives rise to multiple equilibria ranging from "peaceful coexistence" to "trench warfare"?
- Holding the value of continued play fixed, the strategic situation in state $\omega$ is akin to a static game.

Proposition 2 Statewise uniqueness holds provided the outside good is sufficiently unattractive (v large).

- Multiple equilibria must arise from firms' expectations regarding the value of continued play.
Taking the value of continued play as given, the reaction functions intersect once, but there is more than one value of continued play that is consistent with rational expectations.
- Multiplicity is rooted in the dynamics of the model.


## Organizational Forgetting and Multiple Equilibria

- When do multiple equilibria arise?
- In expectation, the "inflow" of know-how into the industry is almost one unit per period, the "outflow" in state $\omega$ is $\Delta\left(\omega_{1}\right)+\Delta\left(\omega_{2}\right)$.
- Consider state $(\omega, \omega)$, where $\omega \geq l$.
- If $1 \ll 2 \Delta(\omega)$, then it is virtually impossible that both firms reach the bottom of their learning curves $\rightarrow$ trench warfare.
- If $1 \gg 2 \Delta(\omega)$, then it is virtually inevitable that both firms reach the bottom of their learning curves $\rightarrow$ peaceful coexistence.
- If $1 \approx 2 \Delta(\omega)$, then primitives do not suffice to tie down the equilibrium $\rightarrow$ multiple equilibria.
Back-of-the-envelope calculation ( $l=15$ and $L=30$ ):

$$
1=2 \Delta(15) \Rightarrow \delta=0.05 \text { and } 1=2 \Delta(30) \Rightarrow \delta=0.02
$$

- Stagewise uniqueness and unidirectional movements through the state space $\rightarrow$ unique equilibrium.
Organizational forgetting makes bidirectional movements possible.

