Dynamic Games: Numerical Methods and Applications

Mini Course, Tel Aviv University, Day 4

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Agenda

- Discussion of problem set.
- Computing all equilibria: Homotopy method.
- Computational burden.
- Open questions.

Learning-by-Doing

- Besanko, D., Doraszelski, U., Kryukov, S. & Satterthwaite, M. (2010) "Learning-by-Doing, Organizational Forgetting, and Industry Dynamics."

- Discrete time, infinite horizon.

- Two firms with potentially different stocks of know-how

$$\omega = (\omega_1, \omega_2) \in \{1, \ldots, L\}^2 = \Omega.$$

- In each period, the timing is as follows:
  - Firms choose prices.
  - One buyer enters the market and makes at most one purchase.
  - Learning-by-doing and organizational forgetting occur and the firms' stocks of know-how change accordingly.

- Law of motion:

$$\omega_n' = \omega_n + q_n - f_n,$$

  where
  - $q_n \in \{0, 1\}$ indicates whether firm n makes a sale with

$$\Pr(q_n = 1) = D_n(p_1, p_2) = \frac{\exp(v - p_n)}{1 + \sum_{k=1}^{2} \exp(v - p_k)};$$

  - $f_n \in \{0, 1\}$ represents organizational forgetting with

$$\Pr(f_n = 1) = \Delta(\omega_n) = 1 - (1 - \delta)^{\omega_n}.$$

Bellman Equation

- Let $V_n(\omega)$ denote the expected NPV to firm n if the current state is ω.
- Firm n's Bellman equation is

$$V_n(\omega) = \max_{p_n} D_n(p_n, p_{-n}(\omega))(p_n - c(\omega_n)) + \beta \sum_{k=0}^{2} D_k(p_n, p_{-n}(\omega)) W_{nk}(\omega),$$

  where

  - $p_{-n}(\omega)$ is the price charged by the other firm;
  - the marginal cost of production is

    $$c(\omega_n) = \begin{cases} \kappa \omega_n^{\eta} & \text{if} \quad 1 \leq \omega_n < l, \\ \kappa l^{\eta} & \text{if} \quad l \leq \omega_n \leq L, \end{cases}$$

    with $\eta = \log_2 \rho$ for a progress ratio of ρ;
  - $\beta \in (0, 1)$ is the discount factor;
  - $W_{nk}(\omega)$ is the expectation of firm n's value function conditional on buyer purchasing good $k \in \{0, 1, 2\}$ (good 0 is outside good).

Bellman Equation

- Continuation values:

$$W_{n0}(\omega) = \sum_{\omega_1'=1}^{L} \sum_{\omega_2'=1}^{L} V_n(\omega') \mathrm{Pr}(\omega_1'|\omega_1, q_1 = 0) \mathrm{Pr}(\omega_2'|\omega_2, q_2 = 0),$$

$$W_{n1}(\omega) = \sum_{\omega_1'=1}^{L} \sum_{\omega_2'=1}^{L} V_n(\omega') \mathrm{Pr}(\omega_1'|\omega_1, q_1 = 1) \mathrm{Pr}(\omega_2'|\omega_2, q_2 = 0),$$

$$W_{n2}(\omega) = \sum_{\omega_1'=1}^{L} \sum_{\omega_2'=1}^{L} V_n(\omega') \mathrm{Pr}(\omega_1'|\omega_1, q_1 = 0) \mathrm{Pr}(\omega_2'|\omega_2, q_2 = 1),$$

where

$$\mathrm{Pr}(\omega_n'|\omega_n, q_n) = \begin{cases} 1 - \Delta(\omega_n) & \text{if} \quad \omega_n' = \omega_n + q_n, \\ \Delta(\omega_n) & \text{if} \quad \omega_n' = \omega_n + q_n - 1, \end{cases}$$

and $\mathrm{Pr}(L|L, q_n = 1) = 1$ and $\mathrm{Pr}(1|1, q_n = 0) = 1$.

Pricing Strategy

- $p_n(\omega)$ is unique solution to FOC:

$$
\begin{aligned}
0 = 1 - (1 - D_n(p_n, p_{-n}(\omega)))\,(p_n - c(\omega_n)) - \beta W_{nn}(\omega) \\
+\beta \sum_{k=0}^{2} D_k(p_n, p_{-n}(\omega)) W_{nk}(\omega).
\end{aligned}
$$

- No closed-firm solution. Solve numerically.

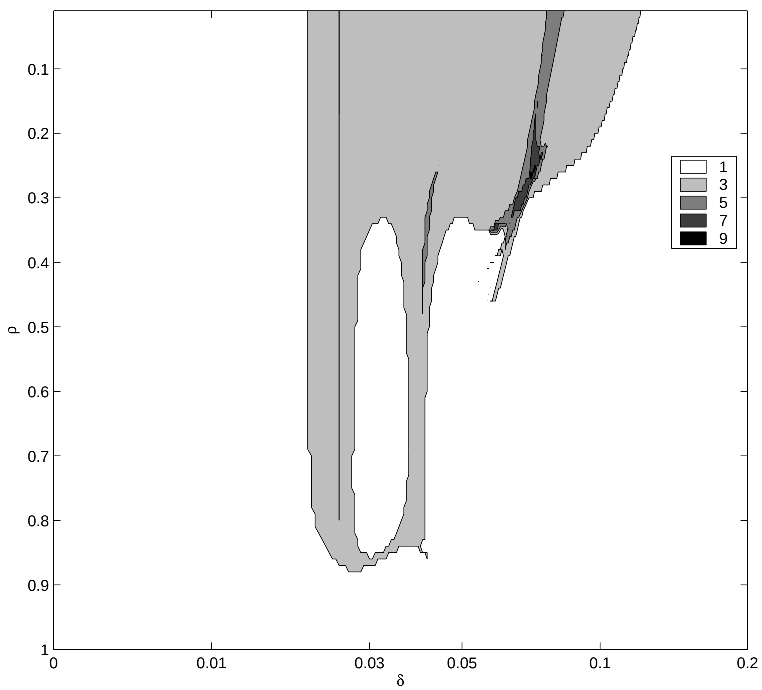
Equilibrium

- Primitives are symmetric.
- Symmetric Markov perfect equilibrium (MPE):
  - Value function $V_1(\omega_1, \omega_2) = V(\omega_1, \omega_2)$ and $V_2(\omega_1, \omega_2) = V(\omega_2, \omega_1)$.
  - Policy function $p_1(\omega_1, \omega_2) = p(\omega_1, \omega_2)$ and $p_2(\omega_1, \omega_2) = p(\omega_2, \omega_1)$.
- Existence in pure strategies is guaranteed (Doraszelski & Satterthwaite 2010), uniqueness is not.
- The goal is to compute the value and policy functions (or, more precisely, $L \times L$ matrices) $\mathbf{V}$ and $\mathbf{p}$.

Multiple Equilibria

Proposition 1 *If organizational forgetting is either absent ($\delta = 0$) or certain ($\delta = 1$), then there is a unique equilibrium.*

Result 1 *If organizational forgetting is neither absent ($\delta = 0$) nor certain ($\delta = 1$), then there may be multiple equilibria.*



Number of equilibria.

Homotopy Method

- Besanko, D., Doraszelski, U., Kryukov, S. & Satterthwaite, M. (2010) "Learning-by-Doing, Organizational Forgetting, and Industry Dynamics."

  Additional reading:

  - Borkovsky, R., Doraszelski, U., & Kryukov, Y. (2010) "A User's Guide to Solving Dynamic Stochastic Games Using the Homotopy Method."

- Show that there are equilibria that the Pakes & McGuire (1994) algorithm cannot compute.

- Propose a homotopy algorithm to trace out the equilibrium correspondence.

Homotopy Method: Learning-by-Doing

- Bellman equation and FOC for state ω are

$$V(\omega) = D_1(\omega)\left(p(\omega) - c(\omega_1)\right) + \beta \sum_{k=0}^{2} D_k(\omega) W_k(\omega),$$

$$0 = 1 - \left(1 - D_1(\omega)\right)\left(p(\omega) - c(\omega_1)\right) - \beta W_1(\omega) + \beta \sum_{k=0}^{2} D_k(\omega) W_k(\omega),$$

  where $D_k(\omega) = D_k(p(\omega), p(\omega_2, \omega_1))$, $k \in \{0, 1, 2\}$.

- The system of $2L^2$ nonlinear equations given by the collection of the above equations for each state $\omega \in \{1, \ldots, L\}^2$ defines a symmetric equilibrium.

Homotopy Method: Learning-by-Doing

- Write the system of $2L^2$ nonlinear equations (Bellman equations and FOCs) as

$$\mathbf{F}(\mathbf{x}, \delta) = 0,$$

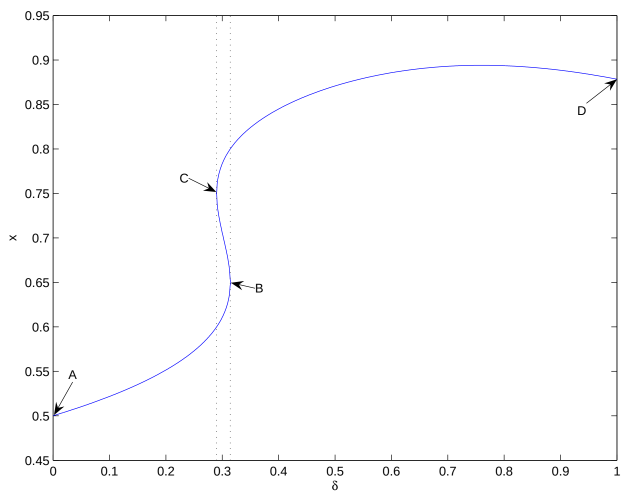
where

$$\mathbf{x} = (V(1,1), \ldots, V(L,L), p(1,1), \ldots, p(L,L)).$$

- The object of interest is the equilibrium correspondence

$$\mathbf{F}^{-1} = \{(\mathbf{x}, \delta) | \mathbf{F}(\mathbf{x}, \delta) = 0\}.$$

- The homotopy algorithm follows a path from the unique equilibrium at $\delta = 0$ to the unique equilibrium at $\delta = 1$.



Equilibrium correspondence $\mathbf{F}^{-1}$ for simple example.

Homotopy Method

- Define a parametric path to be a set of functions $(\mathbf{x}(s), \delta(s))$ such that $(\mathbf{x}(s), \delta(s)) \in \mathbf{F}^{-1}$.

- The conditions that are required to remain "on path" are found by differentiating
$$\mathbf{F}(\mathbf{x}(s), \delta(s)) = 0$$
with respect to s:
$$\sum_{i=1}^{2L^2} \frac{\partial \mathbf{F}(\mathbf{x}(s), \delta(s))}{\partial x_i} x_i'(s) + \frac{\partial \mathbf{F}(\mathbf{x}(s), \delta(s))}{\partial \delta} \delta'(s) = 0.$$

- While there are many solutions, all of them describe the same path.

- One solution obeys the so-called basic differential equations (BDE)
$$y_i'(s) = (-1)^{i+1} \det \left(\left(\frac{\partial \mathbf{F}(\mathbf{y}(s))}{\partial \mathbf{y}} \right)_{-i} \right), \quad i = 1, \ldots, 2L^2 + 1, \qquad (1)$$
where $\mathbf{y}(s) = (\mathbf{x}(s), \delta(s))$ and the notation $(\cdot)_{-i}$ is used to indicate that the ith column is removed from the $(2L^2 \times 2L^2 + 1)$ Jacobian $\frac{\partial \mathbf{F}(\mathbf{y}(s))}{\partial \mathbf{y}}$.

- The BDE reduce the task of tracing out the equilibrium correspondence to solving a system of differential equations.

Homotopy Method: Simple Example

- Consider

$$F(x,\delta) = -15.289 - \frac{\delta}{1+\delta^4} + 67.500x - 96.923x^2 + 46.154x^3$$

with

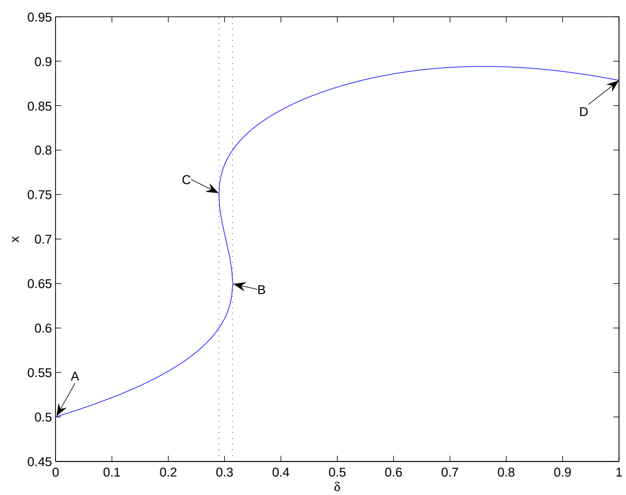
$$\frac{\partial F(x,\delta)}{\partial(x,\delta)} = \begin{pmatrix} 67.500 - 2\cdot 96.923x + 3\cdot 46.154x^2 & -\frac{1-3\delta^4}{(1+\delta^4)^2} \end{pmatrix}.$$

- Basic differential equations:

$$\begin{pmatrix} \frac{dx}{ds} \\ \frac{d\delta}{ds} \end{pmatrix} = \begin{pmatrix} \frac{\partial F(x,\delta)}{\partial \delta} \\ -\frac{\partial F(x,\delta)}{\partial x} \end{pmatrix} = \begin{pmatrix} -\frac{1-3\delta^4}{(1+\delta^4)^2} \\ -67.500 + 2\cdot 96.923x - 3\cdot 46.154x^2 \end{pmatrix}$$

with initial condition $x(0) = 0.5$ and $\delta(0) = 0$.

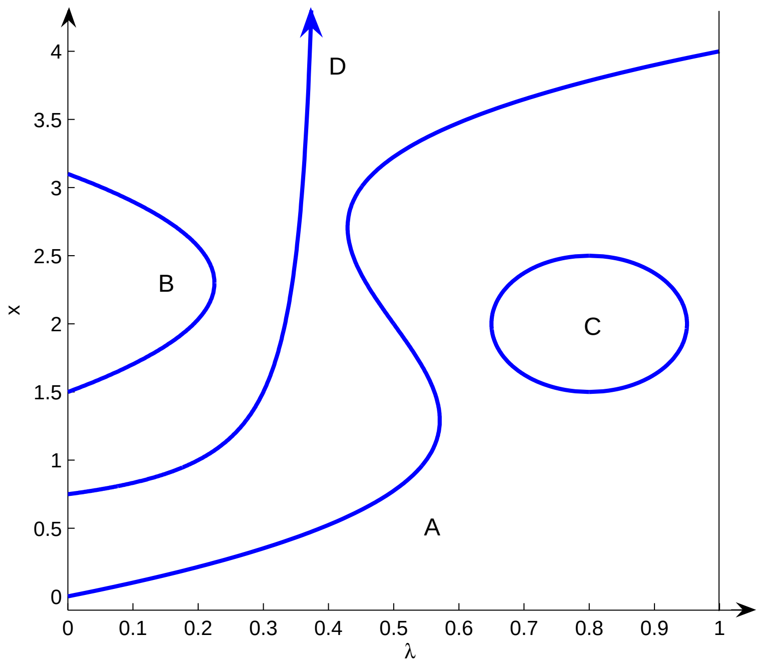
- Solve with e.g. finite-difference methods.

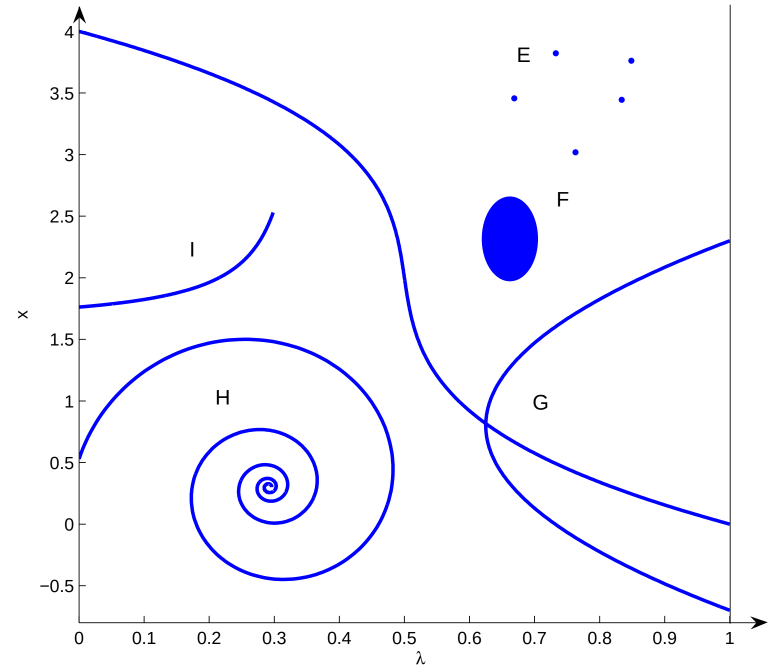


Equilibrium correspondence $\mathbf{F}^{-1}$ for simple example.

Homotopy Method

- The homotopy $\mathbf{F}$ is regular iff $\frac{\partial \mathbf{F}(\mathbf{y})}{\partial \mathbf{y}}$ has full rank at all points in $\mathbf{F}^{-1}$.

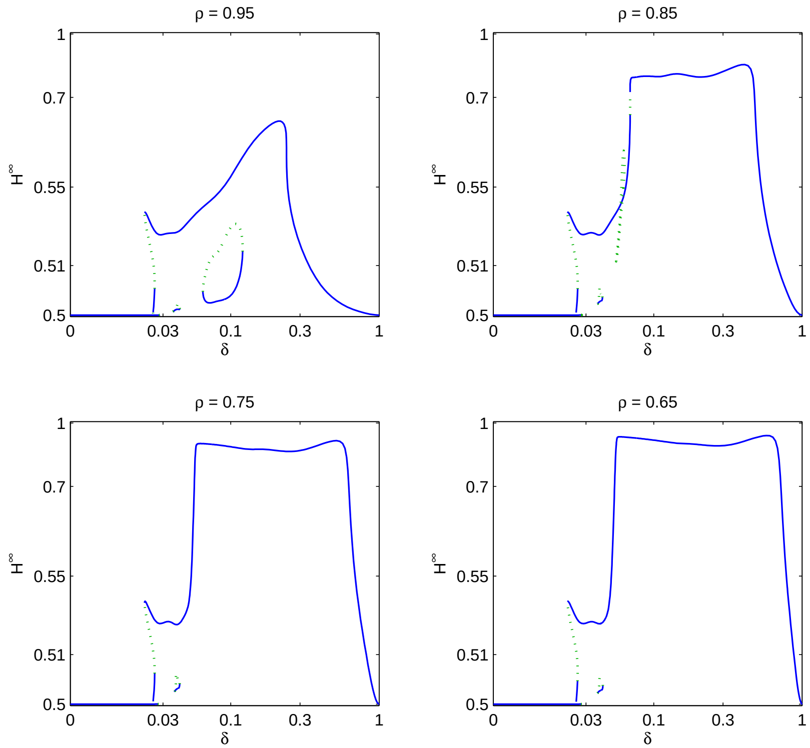




$\mathbf{F}^{-1}$ if $\mathbf{F}$ is regular (left panel) and irregular (right panel).

Equilibrium Correspondence: Learning-by-Doing

Result 2 *The equilibrium correspondence* $\mathbf{F}^{-1}$ *contains a unique path that connects the equilibrium at* $\delta = 0$ *with the equilibrium at* $\delta = 1$*. In addition,* $\mathbf{F}^{-1}$ *may contain (one or more) loops that are disjoint from this "main path" and from each other.*



Limiting expected Herfindahl index H^{∞}. Equilibria with $\varrho\left(\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\Big|_{\delta(s)}\right) < 1$ (solid line) and equilibria with $\varrho\left(\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\Big|_{\delta(s)}\right) \geq 1$ (dotted line).

Pakes & McGuire (1994) Algorithm

- Executes the iteration

$$\mathbf{x}^{l+1} = \mathbf{G}(\mathbf{x}^l), \quad l = 0, 1, 2, \ldots,$$

where, for each state $\omega \in \{1, \ldots, L\}^2$, old guesses for the value and policy of firm 1 are mapped into new guesses as follows:

$$\begin{aligned} p^{l+1}(\omega) &= \arg\max_{p_1} D_1(p_1, p^l(\omega_2, \omega_1))\,(p_1 - c(\omega_1)) \\ &\qquad + \beta \sum_{k=0}^{2} D_k(p_1, p^l(\omega_2, \omega_1)) W_k^l(\omega), \\ V^{l+1}(\omega) &= D_1(p^{l+1}(\omega), p^l(\omega_2, \omega_1))\left(p^{l+1}(\omega) - c(\omega_1)\right) \\ &\qquad + \beta \sum_{k=0}^{2} D_k(p^{l+1}(\omega), p^l(\omega_2, \omega_1)) W_k^l(\omega). \end{aligned}$$

- Let A be an arbitrary matrix and $\varrho(A)$ its spectral radius. Local convergence depends on $\varrho\left(\frac{\partial G(x^*)}{\partial x}\right)$ at the fixed point $x^* = G(x^*)$.

Pakes & McGuire (1994) Algorithm

- "Inbetween" two equilibria that can be computed using the Pakes & McGuire (1994) algorithm, there is one equilibrium that cannot:

  Proposition 2 *If* $\delta'(s) \leq 0$*, then* $\varrho\left(\left.\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\right|_{\delta(s)}\right) \geq 1$.

- Let I denote the $(2L^2 \times 2L^2)$ identity matrix. Then

$$\left.\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\right|_{\delta(s)} = \frac{\partial \mathbf{F}(\mathbf{x}(s), \delta(s))}{\partial \mathbf{x}} + I. \tag{2}$$

- The BDE (1) imply

$$\delta'(s) = \det\left(\frac{\partial \mathbf{F}(\mathbf{x}(s), \delta(s))}{\partial \mathbf{x}}\right).$$

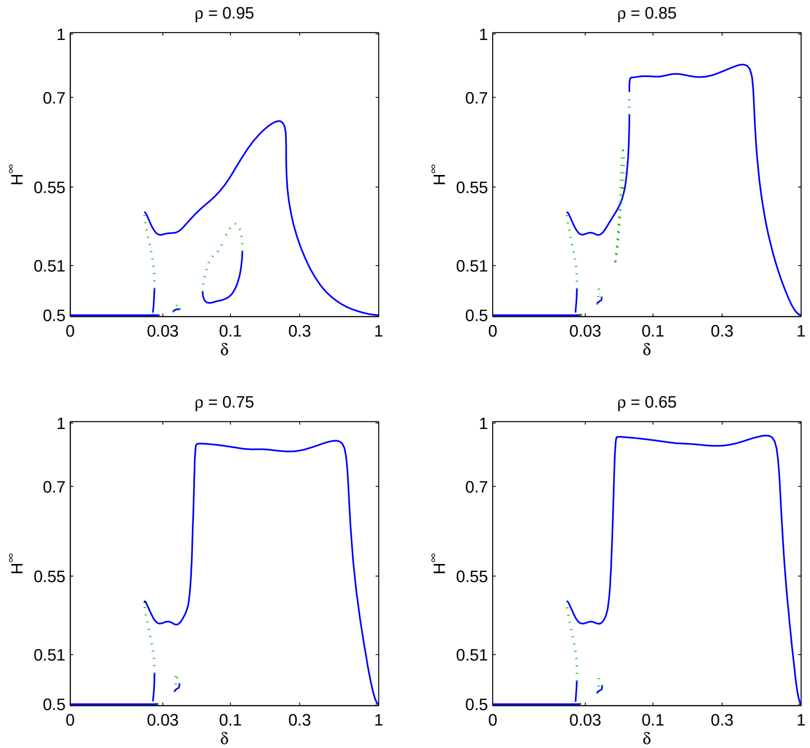
- Since the determinant of $\frac{\partial \mathbf{F}(\mathbf{x}(s),\delta(s))}{\partial \mathbf{x}}$ is the product of $2L^2$ eigenvalues, if $\delta'(s) \leq 0$, then there exists at least one real nonnegative eigenvalue.

- Let A be an arbitrary matrix and $\varsigma(A)$ its spectrum. Then $\varsigma(A + I) = \varsigma(A) + 1$.

- It follows from equation (2) that $\left.\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\right|_{\delta(s)}$ has at least one real eigenvalue equal to or bigger than unity.

Equilibrium Correspondence: Learning-by-Doing

Result 3 *The equilibrium correspondence* $\mathbf{F}^{-1}$ *contains a unique path that connects the equilibrium at* $\delta = 0$ *with the equilibrium at* $\delta = 1$*. In addition,* $\mathbf{F}^{-1}$ *may contain (one or more) loops that are disjoint from this "main path" and from each other.*



Limiting expected Herfindahl index H^{∞}. Equilibria with $\varrho\left(\left.\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\right|_{\delta(s)}\right) < 1$ (solid line) and equilibria with $\varrho\left(\left.\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\right|_{\delta(s)}\right) \geq 1$ (dotted line).

Alternative Approaches to Computing all Equilibria

- If the system of equations is polynomial, then...
  - Judd, K., Renner, P. & Schmedders, K. (2012) "Finding all Pure-Strategy Equilibria in Games with Continuous Strategies."
  - Kubler, F, Schmedders, K. & Renner, P. (2013) "Computing all Solutions to Polynomial Equations in Economics."

- If movements through the state space are undirectional, then...
  - Judd, K. & Schmedders, K. (2004) "A Computational Approach to Proving Uniqueness in Dynamic Games."
  - Judd, K., Schmedders, K. & Yeltekin, S. (2012) "Optimal Rules for Patent Races."
  - Iskhakov, F., Rust, J. & Schjerning, B. (2016) "Recursive Lexicographical Search: Finding all Markov Perfect Equilibria of Finite State Directional Dynamic Games."
  - Iskhakov, F., Rust, J. & Schjerning, B. (2014) "The Dynamics of Bertrand Price Competition with Cost-Reducing Investments."

Sources of Computational Burden

- State space:
  - Suppose that each of N players can be at one of L states.
  - State space has L^N elements.
  - Symmetry reduces exponential to polynomial growth.

- Successor states:
  - Suppose that each of N players can move to one of K states from one period to the next.
  - Expectation over successor states involves K^N terms.

Alleviating the Computational Burden

System of equations:

- Ferris, M., Judd, K. & Schmedders, K. (2007) "Solving Dynamic Games with Newton's Method."

Ergodic set:

- Pakes, A. & McGuire, P. (2001) "Stochastic Algorithms, Symmetric Markov Perfect Equilibrium, and the 'Curse' of Dimensionality."

- Judd, K., Maliar, L. & Maliar, S. (2012) "Merging Simulation and Projection Approaches to Solve High-Dimensional Problems."

State aggregation and interpolation methods:

- Farias, V., Saure, D. & Weintraub, G. (2012) "An Approximate Dynamic Programming Algorithm to Solving Dynamic Oligopoly Models"

- Santos, C. (2012) "An Aggregation Method to Solve Dynamic Games"

- Arcidiacono, P., Bayer, P., Bugni, F. & James, J. (2011) "Sieve Value Function Iteration for Large State Space Dynamic Games."

- Aguirregabiria, V. and Vincentini, G. (2012) "Dynamic Spatial Competition Between Multi-Store Firms."

Alleviating the Computational Burden

Oblivious equilibrium and its extensions:

- Weintraub, G., Benkard, L. & Van Roy, B. (2008) "Markov Perfect Industry Dynamics with Many Firms."
- Weintraub, G., Benkard, L. & Van Roy, B. (2010) "Computational Methods for Oblivious Equilibrium."
- Weintraub, G., Benkard, L. Jeziorski, P. & Van Roy, B. (2008) "Nonstationary Oblivious Equilibrium."
- Benkard, L., Jeziorski, P. & Weintraub, G., (2015) "Oblivious Equilibrium for Concentrated Industries."
- Ifrach, B. and Weintraub, G. (2016) "A Framework for Dynamic Oligopoly in Concentrated Industries."

Alleviating the Computational Burden

Continuous-time stochastic games:

- Doraszelski, U. & Judd, K. (2011) "Avoiding the Curse of Dimensionality in Dynamic Stochastic Games."
- Arcidiacono, P. Bayer, P. Blevins, J. & Ellickson (2016) "Estimation of Dynamic Discrete Choice Models in Continuous Time with an Application to Retail Competition."

Discrete-time stochastic games with alternating moves:

- Doraszelski, U. & Judd, K. (2007) "Dynamic Stochastic Games with Sequential State-to-State Transitions."
- Doraszelski, U. & Escobar, J. (2016) "Protocol Invariance and the Timing of Decisions in Dynamic Games."

Open Questions

What do we know about the general properties of the set of equilibria?

- Doraszelski, U. & Escobar, J. (2010) "A Theory of Regular Markov Perfect Equilibria in Dynamic Stochastic Games: Genericity, Stability, and Purification."

What types of behaviors can arise?

- Besanko, D., Doraszelski, U., Kryukov, Y. & Satterthwaite, M. (2010) "Learning-by-Doing, Organizational Forgetting, and Industry Dynamics."

- Yeltekin, S, Chai, Y. & Judd, K. (2016) "Computing Equilibria of Dynamic Games."

- Doraszelski, U. & Escobar, J. (2012) "Restricted Feedback in Long Term Relationships."

- Balbus, L., Reffett, K. & Wozny, L. (2010) "A Constructive Study of Markov Equilibria in Stochastic Games with Strategic Complementarities."

Open Questions

How can we deal with persistent asymmetric information?

- Fershtman, C. & Pakes, A. (2012) "Dynamic Games With Asymmetric Information: A Framework For Empirical Work."

- Asker, J., Fershtman, C., Jeon, J. & Pakes, A. (2016) "The Competitive Effects of Information Sharing."

- Bernhardt, D. & Taub, B. (2012) "Oligopoly Learning Dynamics."